Assignment #5
Physics 321
Due: Wednesday 04 November 2009

Please answer all questions with complete solutions. All questions are of equal value.

1. Before the midterm, we studied the case of a time-dependent solution to the particle in an infinite well. If the well has width \( L = 1 \) and has two equally possible eigenstates, then the wavefunction can be written

\[
\psi(x,t) = \frac{1}{\sqrt{2}} \left( e^{iE_1 t/\hbar} \phi_1(x) + e^{iE_2 t/\hbar} \phi_2(x) \right)
\]

where as discussed in class \( \phi_n(x) = \sqrt{2} \sin(n\pi x) \) are the stationary eigenstates and \( E_i = \frac{\hbar^2 n^2 \pi^2}{2m} \) are the energies of the states. The probability density for this scenario is

\[
|\phi(x,t)|^2 = \rho(x,t) = \frac{1}{2} \phi_1(x)^2 + \frac{1}{2} \phi_2(x)^2 + \phi_1(x)\phi_2(x) \cos[\Delta E t/\hbar]
\]

which satisfies the normalization constraint \( \int_0^L \rho(x,t) \, dx = 1 \). We graphed this probability density in class, and saw that it “sloshed” back and forth periodically with time through the well. This means that we should be able to observe that behavior in the expression for the probability current.

(a) Compute the probability current for this particle. Is it periodic in time, as expected?

(b) Show that the continuity equation holds for \( \rho(x,t) \) and \( J(x,t) \), i.e. \( \frac{\partial}{\partial t} \rho(x,t) + \frac{\partial}{\partial x} J(x,t) = 0 \).

2. Consider a particle at a potential barrier,

\[
V(x) = \begin{cases} 
0, & x \leq 0 \\
V_0, & x > 0
\end{cases}
\]

where the energy \( E < V_0 \).

(a) Derive expressions for the reflection and transmission probabilities \( R \) and \( T \).

(b) Show, as you would expect, that the reflection probability is \( R = 1 \) [Hint: note that \( R = |\frac{B}{A}|^2 = \left( \frac{B}{A} \right) \left( \frac{B^*}{A^*} \right) \)].

3. A mass \( m \) in a potential \( U(x) = -\frac{1}{2} k x^2 \) experiences simple harmonic motion (e.g. a mass on a spring).
(a) Write the classical Hamiltonian $H$, and then apply canonical quantization \textit{(i.e.} variables become operators) to find the Hamiltonian operator of the corresponding quantum theory.

(b) Write Ehrenfest’s Theorem for the mean values of position and momentum. Double-check that your expression is the quantum equivalent (via the correspondance principle) of the classical equation for the quantity in question.

4. Neutrino oscillations! In preparation for the talk next Friday, let’s do a little problem about neutrinos, which are almost-massless particles that are emitted during nuclear decays. There are three known “flavors” of neutrinos: $\nu_e, \nu_\mu, \nu_\tau$. These correspond to the three types of “leptons” (the electron, the muon, and the tau). It turns out that neutrinos have \textit{two} types of eigenstates: one called the flavor eigenstate, and one called the mass eigenstate. The flavor eigenstates correspond to the physical states of the particles, and the mass eigenstates are what evolve in the Hamiltonian. The flavor eigenstates are \textit{mixtures} of the mass eigenstates, in that they are related by a rotation matrix in Hilbert space: $|\nu_F \rangle = R(\theta) |\nu_m \rangle$.

(a) Consider the case of the two lightest neutrinos, with flavor eigenstates $|\nu_F \rangle = \{|\nu_e \rangle, |\nu_\mu \rangle \}$ and mass eigenstates $|\nu_m \rangle = \{|\nu_1 \rangle, |\nu_2 \rangle \}$. Using the rotation matrix $R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, and the fact that you can express the mass eigenstates as orthogonal vectors, write down a vector-form of the flavor eigenstates.

(b) Verify that the flavor eigenstates are orthogonal.

(c) When the neutrinos have traveled a distance $L$ through space, they will evolve unitarily according to the operator

\[ U = \begin{pmatrix} e^{-im_2^2cL} & 0 \\ 0 & e^{-im_1^2cL} \end{pmatrix} \]

so that $|\nu_m(L) \rangle = U |\nu_m(0) \rangle$. Show that the probability that a $|\nu_e \rangle$ turns into a $|\nu_\mu \rangle$ after travelling a distance $L$ is $P_{\nu_e \rightarrow \nu_\mu} = |\langle \nu_\mu(L) |\nu_e(0) \rangle|^2 = \sin^2 2\theta \sin^2 \left( \frac{\Delta m^2 cL}{4E} \right)$, where $\Delta m^2 = m_2^2 - m_1^2$.

Hint: You will find trig identities useful here, including:

\[ \sin 2\alpha = 2 \sin \alpha \cos \alpha, \quad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \]