A SEQUENCE OF POLYNOMIALS FOR APPROXIMATING ARCTANGENT

HERBERT A. MEDINA

1. Introduction

The Taylor series
\[
\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}
\]
was discovered by the Scotsman James Gregory in 1671 ([B, Ch. 12]). The series converges uniformly to \( \arctan x \) on \([-1, 1]\); thus, we get \( \{T_n(x)\} = \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} \right\} \), the sequence of Taylor polynomials centered at 0 that converges to \( \arctan x \) on \([-1, 1]\).

Like the Taylor polynomials for several other classical functions, e.g., \( \cos x \), \( \sin x \), and \( e^x \), this sequence of polynomials is very easy to describe and work with; but unlike those Taylor sequences with factorials in the denominators of their coefficients, it does not converge rapidly for all “important” values of \( x \). In particular, it converges extremely slowly to \( \arctan x \) when \( |x| \) is near 1. For example, if \( x = 0.95 \), we need to use \( T_{28} \), a polynomial of degree 57, to get three decimal places of accuracy for \( \arctan(0.95) \); if \( x = 1 \), we need to use \( T_{500} \), a polynomial of degree 1001, to get three decimal places for \( \arctan 1 \). Indeed, for \( x \in [0, 1] \), it is easy to show that \( | \arctan x - T_n(x) | \geq \frac{1}{2(2n+3)} \); thus, as \( x \to 1 \), \( T_n(x) \) cannot approximate \( \arctan x \) any better than \( \frac{1}{2(\text{degree } T_n)+4} \). The same is true near \(-1\). It is only fair to note that \( \{T_n\} \) converges to \( \arctan x \) reasonably fast for \( x \) near 0.

In this note we present another elementary, easily-described sequence in \( \mathbb{Q}[x] \) that approximates \( \arctan x \) uniformly on \([0, 1]\) and which does so much more rapidly than the sequence \( \{T_n\} \). Such an approximating sequence provides, via the identities \( \arctan x = -\arctan(-x) = \frac{x}{2} - \arctan\left(\frac{x}{2}\right) \), a method of approximating \( \arctan x \) for all \( x \in \mathbb{R} \). The approximating sequence arises from the family of rational functions \( \left\{ \frac{x^{4m}(1-x)^{4m}}{1+x^2} \right\}_{m \in \mathbb{N}} \).

2. The sequence and its rate of convergence

We begin with an algebraic computation whose proof is easy via induction.
Lemma 1. Define \( p_1(x) = 4 - 4x^2 + 5x^4 - 4x^5 + x^6 \) and \( p_m(x) = x^4(1 - x)^4p_{m-1}(x) + (-4)^{m-1}p_1(x) \) for \( m \geq 2 \). Then
\[
\frac{x^{4m}(1 - x)^{4m}}{1 + x^2} = p_m(x) + \frac{(-4)^m}{1 + x^2}, \text{ for all } m \in \mathbb{N}.
\]

A calculus computation shows that \( x(1 - x) \leq \frac{1}{4} \) on \([0, 1]\). Thus, \( \frac{x^{4m}(1 - x)^{4m}}{1 + x^2} \leq \left(\frac{1}{4}\right)^{4m} \) on \([0, 1]\), and
\[
\int_0^x \frac{t^{4m}(1 - t)^{4m}}{1 + t^2} \, dt \leq \left(\frac{1}{4}\right)^{4m} x \leq \left(\frac{1}{4}\right)^{4m}, \forall x \in [0, 1].
\]

The result of the lemma can be rewritten as \( \frac{x^{4m}(1 - x)^{4m}}{1 + x^2} = p_m(x) - \frac{(-1)^{m+1}4^m}{1 + x^2} \). Thus,
\[
\left| \int_0^x p_m(t) \, dt - \arctan x \right| \leq \left(\frac{1}{4}\right)^{4m}.
\]

Dividing by \((-1)^{m+1}4^m\) and integrating the second term on the left we get
\[
\left| \int_0^x \frac{(-1)^{m+1}4^m}{4^m} p_m(t) \, dt - \arctan x \right| \leq \left(\frac{1}{4}\right)^{5m}.
\]

So
\[
h_m(x) = \int_0^x \frac{(-1)^{m+1}4^m}{4^m} p_m(t) \, dt
\]
defines a sequence in \( \mathbb{Q}[x] \) which converges uniformly on \([0, 1]\) to \( \arctan x \). To get a better sense of the convergence rate, note that \( p_m \) has degree \( 8m - 2 \) and hence \( h_m \) has degree \( 8m - 1 \). In (1) we write \( 4^{5m} = (4^{5/8})^{8m-1+1} \) and summarize our results in Theorem 1.

Theorem 1. For \( m \in \mathbb{N} \), define \( p_m(t) \) as in Lemma 1 and \( h_m(x) \) as in (2). Then
\[
\left| h_m(x) - \arctan x \right| \leq \left(\frac{1}{4^{5/8}}\right)^{\text{degree } h_{m+1}} \text{ for all } x \in [0, 1].
\]

3. Examples, Observations and a Closed-Form Formula

Evaluating \( h_2(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{5x^9}{48} + \frac{x^{10}}{20} - \frac{43x^{11}}{176} + \frac{x^{12}}{4} - \frac{27x^{13}}{208} + \frac{x^{14}}{28} - \frac{3x^{15}}{768} \) at \( x = 0.95 \) and \( x = 1 \) we find that at both points, the approximation to \( \arctan x \) is within \( 2.28 \times 10^{-7} \), better than six decimal places of accuracy with a polynomial of much smaller degree than the Taylor polynomials mentioned in the Introduction. If we consider \( h_7 \), a polynomial of degree 55, (3) guarantees that the approximation on \([0, 1]\) is accurate to within \( 8.47 \times 10^{-22} \).

Thus, \( 4h_7(1) = \frac{506119433541006422355449}{161162819285868563200} \) gives 20 digits of accuracy for \( \pi \).

Like the Taylor polynomials, the \( h_m \) are one-sided approximations. Indeed, it is not hard to see that \( h_m(x) - \arctan x \) is positive when \( m \) is odd and negative when \( m \) is even.

Taylor polynomials are constructed by matching the function and its derivatives at a point. Hermite Interpolating (or osculating) polynomials are constructed by matching
Theorem 2. For any $m \geq 1$, $h_m^{(n)}(0) = \arctan^{(n)}(0)$ and $h_m^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$. Moreover, if $g(x)$ is a polynomial of degree $8m$ such that $g(0) = \arctan 0$, $g^{(n)}(0) = \arctan^{(n)}(0)$ and $g^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$, then $g = h_m$.

Proof. We deal with the $x = 1$ case first. Use (2) and Lemma 1 to note that
\[
\frac{h'_m(x)}{4m} = \frac{(-1)^{m+1}}{4m} p_m(x) = \frac{(-1)^{m+1}}{4m} \left( \frac{x^{4m}(1 - x)^{4m}}{1 + x^2} - \frac{(-4)^m}{1 + x^2} \right)
\]
\[
= \left( \frac{(-1)^{m+1}}{4m} \frac{x^{4m}}{1 + x^2} \right) (1 - x)^{4m} + \frac{1}{1 + x^2}.
\]

(4)

Using $\arctan' x = \frac{1}{1+x^2}$ on the second term and the product rule for differentiation,
\[
\left( f(x)g(x) \right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x),
\]
on the first, we get
\[
h_m^{(n)}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{(-1)^{m+1}}{4m} \frac{x^{4m}}{1 + x^2} \right)^{(n-1-k)} (1 - x)^{4m}^k + \arctan^{(n)}(x).
\]

For $0 \leq k \leq n - 1$, $\left. (1 - x)^{4m}^k \right|_{x=1} = 0$; thus, $h_m^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$.

To prove the assertion at $x = 0$, we can rewrite the first summand in (4) as
\[
\left( \frac{(-1)^{m+1}}{4m} (1 - x)^{4m} \right) x^{4m},
\]
and follow the same steps used above.

If $g$ is a polynomial with the properties stated, then $g - h_m$ is of degree $8m$, and because $g^{(n)}(0) - h_m^{(n)}(0) = 0$ for $0 \leq n \leq 4m$, its first $4m$ coefficients are 0. Hence $g - h_m = x^{4m+1}q$ where $q$ is of degree $4m - 1$. Write $q(x) = \sum_{k=0}^{4m-1} a_k (x-1)^k$. Inductive use of the product rule to compute $(g - h_m)^{(k)}(1)$ shows that $a_k = C_k a_0$ for $1 \leq k \leq 4m - 1$ where $C_k \neq 0$; therefore its use on $(g - h_m)^{(4m)}(1)$ shows $(g - h_m)^{(4m)}(1) = C a_0$ where $C \neq 0$. Thus $a_0 = 0$ and $a_k = 0$ for $1 \leq k \leq 4m - 1$. \hfill \Box

The next lemma is the key in establishing formulas for the coefficients.

Lemma 2. For $m \in \mathbb{N}$, write $\frac{(1 - t)^{4m}}{1 + t^2} = \sum_{j=0}^{4m-2} a_j t^j + \frac{r_m(t)}{1 + t^2}$, where $r_m$ is a polynomial with $\deg(r_m) < 2$. We have
Proof. Write
\[
\frac{(1 - t)^{4m}}{1 + t^2} = \sum_{k=0}^{4m} \binom{4m}{k} (-1)^k t^k
\]
\[
= \sum_{k=0}^{2m} \binom{4m}{2k} t^{2k} + \sum_{k=1}^{2m-1} \frac{4m}{2k+1} t^{2k+1}
\]
\[
SE - SO.
\]
Using \[
\frac{t^{2k}}{1 + t^2} = (-1)^{k+1} \sum_{j=1}^{k} (-1)^j t^{2(j-1)} + \frac{(-1)^k}{1 + t^2}, \text{ for } k \geq 1,
\]
we write
\[
SE = \frac{1}{1 + t^2} + \sum_{k=1}^{2m} \binom{4m}{2k} \left( (-1)^{k+1} \sum_{j=1}^{k} (-1)^j t^{2(j-1)} + \frac{(-1)^k}{1 + t^2} \right).
\]
We collect the polynomial and non-polynomial parts
\[
SE = \sum_{k=1}^{2m} \binom{4m}{2k} \left( (-1)^{k+1} \sum_{j=1}^{k} (-1)^j t^{2(j-1)} \right) + \sum_{k=0}^{2m} \binom{4m}{2k} \frac{(-1)^k}{1 + t^2}.
\]
Because \[
\sum_{k=0}^{2m} \binom{4m}{2k} (-1)^k = (-1)^m 4^m,
\]
the non-polynomial part becomes \[
\frac{(-1)^m 4^m}{1 + t^2}.
\]
We change the order of summation on the polynomial part to get
\[
\sum_{j=1}^{2m} \left( (-1)^j \sum_{k=j}^{2m} \binom{4m}{2k} (-1)^k \right) t^{2(j-1)}.
\]
A similar procedure as that done on \(SE\) shows that
\[
SO = \sum_{j=1}^{2m-1} \left( (-1)^j \sum_{k=j}^{2m-1} \binom{4m}{2k+1} (-1)^k \right) t^{2j-1} + \frac{t}{1 + t^2} \sum_{k=0}^{2m-1} \frac{4m}{2k+1} (-1)^k.
\]
Because the second summand is zero, we have established that
\[
\frac{(1 - t)^{4m}}{1 + t^2} = \sum_{j=1}^{2m} \left( (-1)^j \sum_{k=j}^{2m} \binom{4m}{2k} (-1)^k \right) t^{2(j-1)} + \frac{(-1)^m 4^m}{1 + t^2}
\]
\[
+ \sum_{j=1}^{2m-1} \left( (-1)^j \sum_{k=j}^{2m-1} \binom{4m}{2k+1} (-1)^k \right) t^{2j-1}.
\]
The equality proves both parts of the lemma. □
Combining the results of Lemmas 1 and 2, we see that

\[ p_m(t) = (-1)^m 4^m \sum_{j=1}^{2m} (-1)^j t^{2(j-1)} + \sum_{j=0}^{4m-2} a_j t^{4m+j}, \]  

(5)

where the \( a_j \) are as in Lemma 2. The closed form formula for \( h_m \) follows from (5) and (2).

**Theorem 3.** For \( m \geq 1 \),

\[ h_m(x) = \sum_{j=1}^{2m} \frac{(-1)^{j+1}}{2j-1} x^{2j-1} + \sum_{j=0}^{4m-2} \frac{a_j}{-1)^{m+1} 4^m (4m + j + 1)} x^{4m+j+1}, \]

where \( a_{2i} = (-1)^i+1 \sum_{k=i+1}^{2m} \left( \frac{4m}{2k} \right) (-1)^k \) and \( a_{2i-1} = (-1)^{i+1} \sum_{k=i}^{2m-1} \left( \frac{4m}{2k+1} \right) (-1)^k \).

The theorem makes it easy to use a computer for the computation of the \( h_m \). The author’s website, http://myweb.lmu.edu/hmedina, contains Mathematica code for this computation and more details related to this note.

### 4. Further Remarks and Questions

The keys to the approximating sequence \( \{h_m\} \) are that the family of polynomials \( x^{4m} (1 - x)^{4m}, \ m \in \mathbb{N} \) leaves an integer remainder when divided by \( 1 + x^2 \) and that the members of the family are small for \( x \in [0, 1] \). There are other families of polynomials with this property; is there another simple one that gives a faster approximation to \( \arctan x \)? Is there one with the desirable factorials in the denominator of the error bound?

The results herein were stumbled upon after the author became intrigued by and curious about \( \int_0^1 \frac{x^4 (1-x)^4}{1+x^2} \, dx = \frac{22}{7} - \pi \); that is, \( 4(h_1(1) - \arctan 1) = \frac{22}{7} - \pi \). Is there a simple closed-form formula for \( 4h_1(1) \)? If so, it would provide a sequence in \( \mathbb{Q} \) for approximating \( \pi \). Another, probably-very-difficult, problem is to find an easily-describable sequence \( \{g_n\} \) such that \( 4g_n(1) \) is always a convergent in the continued fraction expansion of \( \pi \).

### References


**Department of Mathematics, Loyola Marymount University, Los Angeles, CA 90045**

E-mail address: hmedina@lmu.edu