2ac. Verify the following identities by block-walking.

(a) \[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.
\]

Solution: We will show that both sides count the number of block walks of length \(n\). Since \(\binom{n}{k}\) counts the number of walks of length \(n\) with exactly \(k\) right turns, and there can be 0, 1, \ldots, \(n\) right turns, so by the addition principle there are \(\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}\) walks of length of \(n\).

Another way to count the walks of length \(n\) is to recognize that on each of the \(n\) blocks the walker has two choices of which way to turn (right or left). Hence there are \(2^n\) such walks.

Since both the right hand side and the left hand side count the number of walks of length \(n\), we have the identity. Q.E.D.

(c) \[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.
\]

Solution: Note that the right hand side counts the number of ways to walk a total of \(m + n\) blocks with exactly \(r\) blocks walked to the right.

We can also construct an \(m + n\) block walk with \(r\) right hand blocks by first choosing \(k\) between 0 and \(r\), and walking \(m\) blocks with \(k\) right turns (in \(\binom{m}{k}\) ways) and then walking \(n\) blocks with exactly \(r - k\) walks to the right. By the product rule, \(\binom{m}{k}\binom{n}{r-k}\) counts the number of \(m + n\) block walks with \(r\) right turns \(k\) of which occur in the first \(m\) blocks. Since \(k\) can equal 0 or 1 or \ldots\ or \(r\), by the addition principle the total number of all such walks is given by \[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.
\]
Since both sides count the same thing, they are equal. Q.E.D.

3d. Verify by the committee selection model.

\[
\sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r}.
\]

**Solution:** Note that the right hand side counts the number of ways to choose a committee of \(m + r\) people from \(m + n\) people (say \(m\) women and \(n\) men).

For the left hand side, let \(k\) be the number of women not on the committee (so that the number of women on the committee is \(m - k\)). Then there are \(\binom{m}{k}\) ways to choose the women not on the committee, and \(\binom{n}{r+k}\) ways to choose the men on the committee. This gives a committee of \(m - k + r + k = m + r\) members. By the product rule, the number of committees of \(m + r\) people with \(k\) women not on the committee is given by \(\binom{m}{k}\binom{n}{r+k}\). Since \(k\) can take on any value from 0 to \(r\), by the addition principle there are

\[
\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r+k}
\]

ways to choose a committee of \(m + r\) people from \(m + n\) people. Thus the left hand side equals the right hand side. Q.E.D.

4c. Verify the identity \(2^n = \binom{n}{0} + \cdots + \binom{n}{n}\) by induction.

**Solution:** We will proceed by induction. To prove the basis step, note that for \(n = 0\), the left hand side is \(2^0 = 1\), while the right hand side is \(\binom{0}{0} = 1\).

To prove the induction step, suppose the result is true for some \(n \geq 0\). That is suppose

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\]

For \(n + 1\), we need to show

\[
\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^{n+1}.
\]
By Pascal’s identity ((3) from text) we have

\[
\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n}{k} + \binom{n+1}{n+1}
\]

\[
= 1 + \sum_{k=1}^{n} \left( \binom{n}{k} + \binom{n}{k-1} \right) + 1
\]

by Pascal’s Identity

\[
= \binom{n}{0} + \left( \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=0}^{n-1} \binom{n}{k} \right) + \binom{n}{n}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k}
\]

\[
= 2^n + 2^n \quad \text{by the induction hypothesis}
\]

\[
= 2^{n+1}
\]

This proves the induction step. Thus the result follows by the principle of mathematical induction.

14cf. By setting \( x \) equal to the appropriate values in the binomial expansion (or one of its derivatives, etc.) evaluate:

(c) \( \sum_{k=0}^{n} 2^k \binom{n}{k} \).

\textbf{Solution:} Let \( x = 2 \) in the binomial theorem. Then

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k = (1 + 2)^n
\]

\[
= 3^n.
\]

(f) \( \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \).

\textbf{Solution:} Note that the first six terms are given by 1, \( \frac{3}{2} \), \( \frac{7}{3} \), \( \frac{15}{4} \), \( \frac{31}{5} \), \( \frac{21}{2} \).

Integrating both sides of the binomial theorem between 0 and 1, we have

\[
\int_{0}^{1} \sum_{k=0}^{n} \binom{n}{k} x^k \, dx = \int_{0}^{1} (1 + x)^n \, dx.
\]
Calculating the integrals, we have

\[
\left[ \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} x^{k+1} \right]^{1} \bigg|_{0}^{1} = \left[ \frac{1}{n+1} (1 + x)^{n+1} \right]^{1}_{0} \quad \text{or}
\]

\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} x^{k+1} = \frac{2^{n+1}}{n+1} - \frac{1}{n+1}.
\]

Thus

\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}.
\]

Note that if we try taking indefinite integrals, we have

\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} x^{k+1} + C_{1} = \frac{1}{n+1} (1 + x)^{n+1} + C_{2}
\]

where \(C_{1}\) and \(C_{2}\) are constants. Rearranging the constants and substituting 1 for \(x\) we have

\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} 1^{k} = \frac{1}{n+1} 2^{n+1} + C
\]

where \(C\) is a constant. To find the constant, we could check the value for \(n = 0\). In this case the left hand side is 1 and the right hand side is \(2 + C\). Thus \(C = -1\). Note that in the case \(n = 1\) then the left hand side is \(\frac{3}{2}\) while the right hand side is 2, telling us that the constant depends on \(n\). This means that we need to undertake a different approach as above.