In this homework we will investigate a generalization of the Riemann Integral called the Riemann-Stieltjes integral of a function. Let \( \alpha \) be a monotonically increasing function (i.e., if \( x < y \) then \( \alpha(x) \leq \alpha(y) \)), and for a partition
\[
P = \{a = x_0, \ldots, x_n\},
\]
define \( \Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \), so that \( \Delta \alpha_i \geq 0 \) for all \( i \). For a function \( f : [a, b] \to \mathbb{R} \), define
\[
U(f, P, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i,
\]
\[
L(f, P, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i,
\]
where \( M_i \) and \( m_i \) are as defined for the standard Riemann Integral. Define
\[
U(f, \alpha) = \inf \{U(f, P, \alpha) \mid P \text{ is a partition of } [a, b]\}
\]
\[
L(f, \alpha) = \sup \{U(f, P, \alpha) \mid P \text{ is a partition of } [a, b]\}.
\]
We now define a function \( f \) to be \textbf{Riemann-Stieltjes }\( \alpha \)-\textbf{integrable} if \( U(f, \alpha) = L(f, \alpha) \), and we define the integral
\[
\int_{a}^{b} f \, d\alpha = U(f, \alpha)
\]
in this case.

Given the above definitions, now answer the following questions.
1. The Riemann Integral is a special case of the Riemann-Stieltjes Integral. For what function $\alpha$ is
\[ \int_a^b f = \int_a^b f d\alpha. \]

2. Let $f : [0, 2] \rightarrow \mathbb{R}$ be the function defined as follows:
\[ f(x) = \begin{cases} 
1 & x \in Q \cap [0, 1] \\
0 & x \in [0, 1] \text{ and } x \notin Q \\
x - 1 & x \in [1, 2]
\end{cases} \]
(a) Is $f$ Riemann Integrable? Why or why not.
(b) Find a monotonic increasing $\alpha : [0, 2] \rightarrow \mathbb{R}$ such that $f$ is Riemann-Stieltjes $\alpha$-Integrable (and prove that it is integrable) with $\int_0^2 f d\alpha = 1/2$
(c) Find an $\alpha : [0, 2] \rightarrow \mathbb{R}$ such that $\int_0^2 f d\alpha = 2$.

3. Show that if $\alpha(0) = 0$ and $\alpha$ is monotonic and continuous, then $f \cdot \alpha$ is Riemann-integrable if and only if $f$ is Riemann-Stieltjes integrable.