You have a great summer job in a research laboratory with a group investigating the possibility of producing power from fusion. The device being designed confines a hot gas of positively charged ions, called plasma, in a very long cylinder with a radius of 2.0 cm. The charge density of the plasma in the cylinder is $6.0 \times 10^{-5}$ C/m$^3$. Positively charged tritium ions are to be injected into the plasma perpendicular to the axis of the cylinder in a direction toward the center of the cylinder. Your job is to determine the speed that a tritium ion should have when it enters the plasma cylinder so that its velocity is zero when it reaches the axis of the cylinder. Tritium is an isotope of hydrogen with one proton and two neutrons.

To solve this problem (find the necessary speed of the tritium) we will use Gauss’ Law as well as the idea of conservation of energy.

Here’s a sketch of our plasma:

The radius of the cylinder is $R = 0.02$ m.
The plasma has a charge density of $\rho = 6.0 \times 10^{-5}$ C/m$^3$.

The tritium enters the plasma along a radial trajectory.

The tritium has a charge of $q = +1.6 \times 10^{-19}$ C (net charge of one proton), and a mass of $m = 5.023 \times 10^{-27}$ kg.

The basic course of action will be the following:
1. Find an expression for the electric field inside the cylinder using Gauss’ Law
2. Use this electric field expression to find the potential energy of the system.
3. Employ the concept of conservation of energy to finally solve for the initial speed necessary to just reach the center.

Some assumptions we’ll need to follow this plan:
- Assume that the tritium starts on the outer edge of the plasma (this is the initial position).
- Assume that the charge density of the plasma is uniform.
- Assume that the tritium particle does not effect the plasma (in other words the charge density is fixed).
• Assume that there are no collisions or other interactions besides the electrostatic forces.
• Take “long cylinder” to mean nearly infinite. That way we can ignore the effects of the ends on the electric field near the middle (where the tritium is injected).

To first find the electric field inside the plasma we must use Gauss’ Law and a cylinder for a Gaussian surface. This cylinder will be of radius $r$ and length $L$.

The charge inside of this Gaussian surface will be the charge density times the volume:

$$Q = \rho \pi r^2 L$$

The flux through this surface is simply given by the electric field times the surface area of the curved portion of the cylinder. This is due to the symmetry of the problem and the fact that the electric field is always parallel to the normal vector on this surface (and perpendicular to the normal on the ends)

$$\Phi = E 2\pi r L$$

We combine these two expressions with Gauss’ law to find the electric field:

$$E = \frac{\rho r}{2\varepsilon_o}$$

To find the potential energy stored in the system we will use this field and the charge of the tritium (together they determine the force on the particle). You should ask yourself- does the potential energy get bigger or smaller as the particle moves to the center of the plasma? In other words, will the particle slow down (loosing kinetic energy or increasing potential energy) or speed up (decreasing potential energy)?

$$U_f - U_i = \Delta U = -\int_R^0 Fdr = -\int_R^0 qEdr = -\int_R^0 q\rho r dr = \frac{q\rho R^2}{2\varepsilon_o}$$

The potential energy has increased as we’ve moved toward the center (against the electrostatic force) much as if we were going up a hill.

We can now use conservation of energy. We will set the initial potential energy equal to zero (at the plasma’s edge). Also, remember that the final kinetic energy is equal to zero.
\[ E_i = E_f \]
\[ U_i + K_i = U_f + K_f \]
\[ 0 + \frac{1}{2}mv^2 = \frac{q\rho R^2}{4\varepsilon_o} + 0 \]

So, we can now solve for the initial speed, \( v \).

\[ v = \sqrt{\frac{q\rho R^2}{2m\varepsilon_o}} \]

So, \( v = 2.1 \times 10^5 \text{ m/s} \). This could be a reasonable value- it’s not greater than the speed of light and greater than a snail’s pace. Also, the units work out just fine in the final expression.
As you so often do, you find yourself reflecting on past physics discussions, trying to make sense of them in light of your new knowledge. (You do this, right?!?) The physics demo that is in your mind today is the one that we did a few weeks ago which had a charged balloon “picking up” small pieces of paper. The balloon, which is a typical one of radius ~10cm, was rubbed on hair, then brought near the pieces. When the balloon got within 10cm of the pieces, they jumped up to the balloon. You decide to apply what you now know about electric fields to this situation and determine the amount of charge on the balloon. You estimate that the pieces of paper where about 3mm in radius and 0.1mm thick (typical 20lb paper).

(Alternatively,… you perform the same experiment with pieces of aluminum foil, 1mm radius and 0.01mm thick, and see that the same balloon only needs to be 20cm away from the foil pieces before they jump.)

As mentioned in class, there are essentially two different problems here, one with paper and one with foil. Since one is an insulator and the other is a conductor, there are different conditions and assumptions at play in each version. First I present the foil version, then the paper version.

Foil

Why are the pieces picked up? Are they charged? Probably not; even if we did say that they were charged, we would have no way of determining the net charge. No, it is more likely that the pieces are neutral. So,… why are the pieces picked up? A picture might help.

Notice that in the (enlarged) foil, the top is positive and the bottom is negative. This separation of charge happens because the piece is made from a conductor, which allows the electrons to move freely.

But, why doesn’t the repulsive force on the bottom charges negate the attractive force on the top charges? The bottom charges are further away, so they see a slightly weaker electric field from the balloon. Even though the field is only slightly weaker, it is enough to ensure that the attractive force is greater than the repulsive one.

We still need to figure out how many positive and negative charges end up on the foil’s surface.

Now that we have an intuitive feel for what is going on, let’s outline the relevant physics ideas.

- The foil will be treated as an ideal conductor. This means that the total electric field inside of the foil is zero.
- The surface charge on the foil can be found if we treat it as a flat disk that is approximately infinite.
• The balloon will create an electric field that will interact with the foil. Since the balloon can be modeled as a sphere, we will use the expression for spherical charge distributions.

\[ R = 0.3 \text{m, distance from the balloon’s center to the top of the foil} \]
\[ Q, \text{ balloon’s unknown charge} \]
\[ \sigma, \text{ foil’s unknown surface charge density} \]
\[ \rho = 2700 \text{ kg/m}^3, \text{ density of aluminum} \]
\[ h = 10^{-5} \text{m, foil thickness} \]
\[ A, \text{ surface area of the foil} \]

First, let’s employ the \( \sum E = 0 \) condition:

\[ E_{\text{balloon}} + E_{\text{foil}} = 0 \]

Where the balloon’s electric field is:

\[ E_o = \frac{kQ}{R^2} \]

Treating the pieces as flat disks, allows us to find the surface charge density in terms of the balloon’s field.

\[ \sigma = \varepsilon_o E_o = \frac{\varepsilon_o kQ}{R^2} \]

This means that the total charge on the top and bottom surfaces is:

\[ q = \sigma A \]

So,

\[ q = \varepsilon_o E_o A = \varepsilon_o \left( \frac{kQ}{R^2} \right) A \]

(Notice that we are treating the balloon’s field to be uniform at the foil. This is okay for this part of the problem, but when we consider the forces, we will need to be more precise and find separate fields at the top and bottom of the foil.)

Now, we can consider the forces on the foil. There are three- gravity, attractive electric force on the top, and a repulsive electric force on the bottom. If we say that the net force is zero, then the foil is in equilibrium. To get the foil rise toward the balloon we would only need to slightly increase the forces. Essentially, we are finding the limiting case for the balloon’s charge. If the charge is just a tiny bit bigger, then the pieces rise upward.
Net force:
\[ \sum F = E_{o,\text{top}}q - E_{o,\text{bottom}}q - mg = 0 \]
\[ \Rightarrow E_{o,\text{top}}q - E_{o,\text{bottom}}q = mg = \rho Ahg \]

Notice that here we are keeping track of the balloon’s electric field at the top and bottom separately. In the previous step we ignored this variation:

\[ E_{o,\text{top}} = \frac{kQ}{R^2} \]
\[ E_{o,\text{bottom}} = \frac{kQ}{(R + h)^2} \]

(Also, notice that in the force equation, up is taken to be positive direction.)

Now, we substitute in our expressions for charge and electric field into the force equation:

So,
\[ \left[ \varepsilon_0 \left( \frac{kQ}{R^2} \right) A \left( \frac{kQ}{R^2} - \frac{kQ}{(R + h)^2} \right) \right] = \rho Ahg \]

\[ \varepsilon_0 \left( \frac{kQ}{R} \right)^2 \left( \frac{1}{R^2} - \frac{1}{(R + h)^2} \right) = \rho hg \]

Solve for \( Q \),
\[ Q = \sqrt{\frac{\rho hg}{\varepsilon k^2 \left( \frac{1}{R^2} - \frac{1}{(R + h)^2} \right) }} \]

\( Q \approx 2 \times 10^{-4} \text{ C} \)

This seems plausible, as it is still a fraction of a Coulomb.
**Paper**

The physics for this solution is exactly the same as the foil’s except for one step. There we had employed the E=0 condition inside the foil, but since paper is an insulator, this will not be the case. However, there is still a specific condition on the field inside the paper that will allow us to once again related the surface charge to the balloon’s field.

The electric field inside of the paper will now be:

$$E_{inside} = \frac{E_o}{\kappa}$$

where $\kappa$ is the dielectric constant for paper ($\kappa=3.7$)

The electric field produced by the induced surface charges is:

$$E_{inside} = \frac{E_o}{\kappa} = E_o - E_{surface} \Rightarrow E_{surface} = E_o \left( 1 - \frac{1}{\kappa} \right)$$

Just as before we’ll treat the piece and the charges on the surface as a flat disk configuration:

$$\sigma = \varepsilon_o E_{surface} = \varepsilon_o E_o \left( 1 - \frac{1}{\kappa} \right)$$

From here on out, except for the different numerical values, the process is the same:

External field, from the balloon:

$$E_o = \frac{kQ}{R^2}$$

So, the charge on the paper surfaces is:

$$q = \varepsilon_o E_o \left( 1 - \frac{1}{\kappa} \right) A = \varepsilon_o \left( \frac{kQ}{R^2} \right) \left( 1 - \frac{1}{\kappa} \right) A$$

The net force is:

$$\sum F = E_{o,\text{top}} q - E_{o,\text{bottom}} q - mg = 0$$

$$\Rightarrow E_{o,\text{top}} q - E_{o,\text{bottom}} q = mg = \rho Ahg$$
with
\[ E_{o,top} = \frac{kQ}{R^2} \]
\[ E_{o,\text{bottom}} = \frac{kQ}{(R + h)^2} \]

So,
\[ \left[ \varepsilon_0 \left( \frac{kQ}{R^2} \right) \left( 1 - \frac{1}{\kappa} \right) A \right] \left( \frac{kQ}{R^2} - \frac{kQ}{(R + h)^2} \right) = \rho Ahg \]
\[ \left[ \varepsilon_0 \left( \frac{kQ}{R} \right)^2 \left( 1 - \frac{1}{\kappa} \right) \right] \left( \frac{1}{R^2} - \frac{1}{(R + h)^2} \right) = \rho hg \]

Solving for \( Q \),
\[ Q = \sqrt{\frac{\rho hg}{\varepsilon_0 k^2 \left( 1 - \frac{1}{\kappa} \right) \left( \frac{1}{R^2} - \frac{1}{(R + h)^2} \right)}} \]

here:
\( R = 0.2 \text{m}, \) distance from the balloon’s center to the top of the paper
\( Q, \) balloon’s unknown charge
\( \rho = 750 \text{ kg/m}^3, \) density of paper
\( h = 10^{-4} \text{m}, \) paper thickness

\( Q \approx 4.5 \times 10^{-5} \text{C} \)