A Simple Model for Predicting Sprint Race Times Accounting for Energy Loss on the Curve

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Abstract
The mathematical model of J. Keller for predicting World Record race times, based on a simple differential equation of motion, predicted quite well the records of the day. One of its shortcomings is that it neglects to account for a sprinter's energy loss around a curve, a most important consideration particularly in the 200 m to 400 m range. An extension to Keller's work is considered, modeling the aforementioned energy loss as a simple function of the centrifugal force acting on the runner around the curve. Theoretical World Record performances for indoor and outdoor 200 m are discussed, and the use of the model at 300 m is investigated. Some predictions are made for possible 200 m outdoor and indoor times as run by Canadian 100 m WR holder Donovan Bailey, based on his 100 m final performance at the 1996 Olympic Games in Atlanta.

1 Introduction

In 1973, mathematician J. Keller [1] proposed a model for predicting World Record (WR) race times based on a simple least-square fit of the records of the day. The fit was quite good, and provided a simple tool for gauging possible optimal performances in races, based on results from others. Keller's model was limiting in the sense that it could only "in reality" predict possible records of linear races, with no consideration for those run on curves.

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For distance races (over 400 m), the impact of running the curve is negligible. When the race speeds are much higher, though, the curve contributions cannot be left out.

Recent WR performances in athletics have prompted various new studies based on Keller's work. Tibshirani [2] introduces a more realistic energy loss model for sprinting, accounting for the sprinter's actual velocity curve. Still, though, the curve of the track is not considered; this is mentioned in [2], but no solution is offered. The following work will formulate a simple model to account for energy loss around the curve, and predict possible WR performances accordingly, using data obtained from a least-square fit of contemporary short sprint records. Both outdoor races, as well as indoor competitions, are discussed. As a practical example, the 100 m WR sprint race of Donovan Bailey (Canada) is used as empirical data to further determine the validity of the model for predicting 200 m sprint times. A brief discussion of indoor 300 m records is offered. The possibility of using such a model as a training tool for athletes and coaches in considered.

2 The Keller Model

Although mathematical models for running were first introduced by A. V. Hill [3] in the mid 1920s, it was J. Keller who formulated a model to predict possible WR performances [1], based on the notion that the speed and energy loss of a human can be determined by certain key variables. In its simplest form, the Keller (or Hill-Keller) model is a solution to the simple equation of motion

$$\dot{v}(t) = f(t) - \tau^{-1} v(t) \ , \tag{1}$$

Here, $f(t)$ is the force per unit mass exerted by the runner, and τ is a decay constant which models internal (physiological) resistances felt by the runner. The differential equation (1) is solved subject to the constraint $v(0) = 0$, and also bearing in mind that $f(t) \leq f$ (*i.e.* the runner can only exert so much force). The length of the race d can be calculated as

$$d = \int_0^T dt\, v(t) \ , \tag{2}$$

and the time T to run the race can be obtained for a particular velocity curve $v(t)$ over d. An additional constraint is that the power $f(t) \cdot v(t)$

must equal the rate of internal energy supply (cellular oxygen replacement, anaerobic reactions, *etc...*),

$$\frac{dE}{dt} = \sigma - f(t)v(t) \ , \tag{3}$$

with σ a physiological term representing the body's energy balance. This is coupled with the initial condition $E(0) = E_0$, as well as the non-negativity of $E(t)$ ($E(t) \geq 0$) [1].

By variational methods, it was determined [1] that the optimal strategy for short sprints ($d < 291m = d_{crit}$) is for the runner to go all-out for the duration of the race. That is, $f(t) = f$. Hence, $v(t)$ and d can be calculated simply as

$$\begin{aligned} v(t) &= f\tau(1 - e^{-t/\tau}) \ , \\ d &= f\tau^2 \left(T/\tau + e^{-T/\tau} - 1\right) \ . \end{aligned} \tag{4}$$

For races of $d > d_{crit}$, the runner should chose a different optimization strategy. Keller determined the parameters by performing a nonlinear least-squares fit to the track records of the day, obtaining [1]

$$\begin{aligned} \tau &= 0.892 \text{ s} \\ f &= 12.2 \text{ m/s}^2 \\ \sigma &= 9.83 \text{ cal/(kgs)} \\ E_0 &= 575 \text{ cal/kg} \end{aligned} \tag{5}$$

Keller [1] determined the optimal times (and hence WRs) for short sprints, and found: 50 m, 5.48 s; 60 m, 6.40 s; 100 m, 10.07 s; 200 m, 19.25 s; 400 m, 43.27 s. Although 400 m is beyond the short sprint category, this time is cited because of its incredible approximation to the current record (43.29 s, Harry "Butch" Reynolds, 1988). Andre Cason's (USA) 6.41 s 60 m WR is also surprisingly close.

3 Tibshirani's Extension

It is somewhat unrealistic to believe that a sprinter can actually apply a constant force for the duration of a race. This being said, it seems logical to assume the force $f(t)$ decreases with time. In [2], a linear decrease was chosen, $f(t) = f - ct$, where $c > 0$. In this case, the equations of motion become

$$\begin{aligned} v(t) &= k - ct\tau - ke^{-t/\tau} \ , \\ D(t) &= kt - \frac{1}{2}c\tau t^2 + \tau k(e^{-t/\tau} - 1) \ , \end{aligned} \tag{6}$$

with $k = f\tau + \tau^2 c$.

More complex time dependences could equivalently be chosen (for example, it might be more appealing to chose a time dependence of the form $f(t) = f\exp(-\beta t)$), but for the purposes of this study, the linear one will suffice.

3.1 Accounting for reaction time

The values in (5) were calculated without consideration of reaction time on the part of the athlete. The IAAF sets the lowest possible reaction time by a human being to be $t_{reac} = 0.100$ s; any sprinter who reacts faster than this is charged with a false start. These times generally do not drop below +0.130 s, and in general register around +0.150 s (the average reaction time for the 100 m and 200 m finals at the 1996 Olympic Games was roughly +0.16 s). Granted, the ability to react quickly is an important strategy, and obviously one which cannot really be fit into a model. At the 1996 Olympic Games, American sprinter Jon Drummond registered a reaction time of +0.116 s (100 m, round 2, heat 2), and in the semifinal defending champion Linford Christie (GBR) reacted in +0.124 s [5]. Such quick reactions tend to be more a result of anticipating the starter's gun, though, rather than purely electrophysiological manifestations.

4 Physical Meaning of the Parameters

Although mathematical in origin, it is reasonable to hypothesize what might be the physical interpretation of the parameters (f, τ, c). Clearly, f is a measure of the raw acceleration capability of the sprinter, while $f\tau$, having units of ms^{-1}, is a representation of velocity. In fact, this is the maximum velocity which the sprinter is capable of attaining (in the Keller model only; in the Tibshirani extension, the expression is slightly more complicated).

The variable c must have units of f/t, hence m s^{-3}. Ideally, this is the time rate of change of the runner's output, and can be thought of as a measure of muscular endurance. The full implications of τ are unclear, but due to the nature of the equation of motion, and keeping in mind the initial conjecture of Keller that it be a function of internal resistances, one could hypothesize τ to be some type of measure of such elements as flexibility, leg turnover rate, anaerobic responses, and so forth.

While not necessarily representative of any *exact* physical quantity, these parameters may have some physical analogue. The mechanics of sprinting is far more complicated than the model suggests. However, the mere fact that these models can predict race times with surprising accuracy indicates that perhaps they can be of some use in training. One could imagine that a determination of the set (f, τ, c) for athletes can help to gear workouts toward specific development (power, endurance, and so forth). Further investigation of the consistency of the model for various athletes might considered.

5 200 m races: Adjusting for the Curve

It is the opinion of this author that the way a sprinter handles the curve portion of a race, in particular a 200 m, cannot be discounted. Exactly how this should be taken into consideration is unknown, as there are surely various factors (both physical and physiological) which must be addressed. The only physical difference between straight running and curve running is obviously the effects of centrifugal forces on the sprinter. One can assume that a sprinter's racing spikes provide ample traction to stop outward translational motion, so this is not a concern. To compensate for the rotational effects (torques), the sprinter leans into the turn. This is not constant during the race; greater speeds require greater lean. However, the degree of lean is limited by the maximum outward angle of flexion of the ankle. Furthermore, one would think that maximum propulsive efficiency would not be generated at this extreme limit.

So, a curve model is not a trivial one to construct. However, based on the physical considerations alone, let us assume that the effect will manifest itself as a centrifugal term in the equation of motion. Since this is normal to the forward motion of the sprinter, we can rewrite (1) as

$$f(t)^2 = \left(\dot{v}(t) + \tau^{-1}v(t)\right)^2 + \lambda^2 \frac{v(t)^4}{R^2}, \tag{7}$$

The term $\lambda < 0$ has been added to account for the fact that a sprinter does not feel the full centrifugal force resulting from his angular velocity. This seems to be the simplest choice, at least for a first approximation to the correction. Clearly, the Hill-Keller model is regained in the limit $R \to \infty$ (alternatively $\lambda \to 0$).

The radius of curvature R can have two distinct sets of values, depending on whether the competition is indoor or out,

$$\begin{aligned} R_{\text{outdoor}} &= \left[\frac{100}{\pi} + 1.25(p-1)\right] \text{ m} , \\ R_{\text{indoor}} &= \left[\frac{50}{\pi} + 1.00(p-1)\right] \text{ m} , \end{aligned} \tag{8}$$

Here, p is the lane number, and the factors 1.25 (outdoor) and 1.00 (indoor) have been chosen as suitable representations of IAAF regulation lane widths, according to the following standards [4]:

- Outdoor: 400 m in the inside lane, comprised of two 100 m straights, and two 100 m curves of fixed radius. Lane widths can range between 1.22 and 1.25 m, and are separated by lines of width 5 cm.
- Indoor: 200 m in the inside lane (two 50 m straights, and two 50 m curves). The lanes (4 minimum, 6 maximum) should be between 0.90 m to 1.10 m in width, separated by a 5 cm thick white line. The curve may be banked up to 18°, and should have a radius between 11 m and 21 m. The radius need not be constant.

Solving Equation (7) for $\dot{v}(t)$, with $f(t) = f$, one obtains

$$\dot{v}(t) = -\tau^{-1}v(t) + \sqrt{f^2 - \lambda^2 \frac{v(t)^4}{R^2}} \ . \tag{9}$$

Equivalently, for Tibshirani's more realistic model ($f(t) = f - ct$), Equation (9) becomes

$$\dot{v}(t) = -\tau^{-1}v(t) + \sqrt{(f-ct)^2 - \lambda^2 \frac{v(t)^4}{R^2}} \ . \tag{10}$$

Because of a current lack of necessary empirical sprint data, the value of λ can only be estimated.

Differential equations of the form (9), (10) are not trivial to solve, as they yield no explicit solutions for $v(t)$. However, such are easily solved by numerical methods. This was performed on the MAPLE V Release 4 mathematical utility package, which uses a fourth-fifth order Runge-Kutta method.

The race distance d traversed around the curve in time T can be calculated analogously to Equation (2),

$$\begin{aligned} d &= d_c + d_s \\ &= \int_0^{t_1} dt\, v_c(t) + \int_{t_1}^{T} dt\, v_s(t) \ , \end{aligned} \tag{11}$$

with $v_c(t)$ the solution to Equation (10), and $v_s(t)$ the velocity as expressed in Equation (6), solved for the boundary condition $v_c(t_1) = v_s(t_1)$. Here, t_1 is the time required to run the curved portion of the race (distance d_c), the integral form of which is evaluated numerically, based on the method of calculation stated for $v_c(t)$.

By using Keller's parameters (5), we can adjust his original prediction of 19.25 s to account for the curve. In fact, as an aside, it should be mentioned that the record of 19.5 s as indicated in [1] is in fact the straight-track record of Tommie Smith, from 1966 [4]. With this in mind, we can apply the result of (9), coupled with (11), to obtain an outdoor curved track WR estimate. For an assumed $\lambda^2 = 0.60$ (see section 9.1 for a discussion on choice of its value):

$$\begin{aligned} v_{100} &= 10.66\,\mathrm{m/s}\ , \\ t_{100} &= 10.24\,\mathrm{s}\ , \\ t_{200} &= 19.46\,\mathrm{s}\ . \end{aligned} \tag{12}$$

The IAAF notes that times run on curves were estimated to be 0.3 to 0.4 s slower than straight runs [4]. These results would tend to agree with this assertion.

6 New Model Parameters for Modern World Records

The parameters (5) are more than likely out of date, as they were calculated by fitting records almost 25 years old [1]. Also, these were fitted for a model which does not accurately model the velocity curves of sprinters. For example, a 100 m runner's velocity is not strictly increasing, but rather peaks between 40 and 60 m. Table 1 lists the sprint WRs as of March 1997, from 50 m to 400 m [4, 5].

New parameters (f, τ) and (f, τ, c) have been obtained (by a similar method to that of Keller [1]) from the four straight-track sprint WRs (50 m, 55 m, 60 m, and 100 m), and are listed in (13,14). These reproduce the short sprint times quite well (Table 2). The equations 6 were fit by a nonlinear least-squares method, using the Statistical Analysis Software (SAS) package. In all cases considered here, appropriate convergence criteria were met after several iterations (using the Gauss-Newton method). The upper and lower asymptotic 95% confidence levels for each fit herein are given.

Aside from the 100 m WR (where the reaction time is known, $t_{react} = +0.174$ s [8]), a (perhaps liberal) reaction time of +0.16 s has been assumed. By using the indoor races to calculate parameters, one is inherently removing the possibility of wind-assisted times. This has not been done in the case of the 100 m WR (where the wind-reading was +0.7 m/s [8]), which may provide some source of error[1]

$$\begin{aligned} f &= 10.230 \ \text{m/s}^2 \\ \tau &= 1.147 \ \text{s} \end{aligned} \tag{13}$$

with (lower, upper) asymptotic 95% confidence levels of $f = (10.060, 10.399)$, $\tau = (1.124, 1.170)$, and

$$\begin{aligned} f &= 9.596 \ \text{m/s}^2 \\ \tau &= 1.274 \ \text{s} \\ c &= 0.058 \ \text{m/s}^3 \end{aligned} \tag{14}$$

with (lower, upper) asymptotic 95% confidence levels of $f = (8.290, 10.901)$, $\tau = (0.981, 1.567)$, $c = (-0.065, 0.180)$.

In light of the discussions of Tibshirani's extension with relation to observed velocity curves, the parameters (13) are cited only for comparison

[1]Prictchard [6] offers a simple method of accounting for wind assistance and drag. Making use of his work, one finds that in fact Donovan Bailey's 9.84 s WR corrects to a 9.88 s still-wind reading. This is surpassed by Frank Fredericks 9.86 s run with a wind reading of −0.4 m/s, which adjusts to roughly 9.84 s [7]. So, if we account for wind contributions, a similar time is obtained anyway.

with older values (although predictions using (13) are offered in Table 4, as a comparison to Keller's results). Otherwise, this work will use only the parameters of (14).

7 Predicting the 200 m World Record

By a straight application of the model as described above, it is possible to obtain predicted WR times for the 200 m sprint. In addition, it seems logical to obtain predictions for indoor 200 m races, as well, where the dynamics of curve sprinting should be more apparent. For outdoor performances, $d_c = 100$ m in (11), and $d_s = 100$ m, and this is the same for all eight lane choices ($p = [1, 8]$). Recall that d_c is not the *curve length* for all lanes, only the distance run on the curve. For indoor races, the total distance is calculated by

$$d = d_{c1} + d_s + d_{c2} + d_s \ , \tag{15}$$

where $d_{c1,2}$ depend on the lane choice. Since standard indoor tracks are 200 m in lane 1, it follows that $d_{c1} = d_{c2} = d_s = 50$ m. The radius obviously increases for subsequent lanes, and using (8), one obtains $d_{c1} = 40.58$ m and $d_{c2} = 59.42$ m. The latter value is the total length of the curved portion of lane 4, while the former is the distance run after the stagger.

For all tables, unless otherwise indicated the times listed will be raw (*i.e.* minus reaction time). Only the final race times include reaction, as indicated in the column headings.

7.1 Outdoor 200 m

Calculations using various values of increasing λ (λ^2) are detailed in Table 5 and Table 6. For outdoor races (Table 5), a λ^2 range of 0.50 to 0.80 has been used. Before the 1996 Olympic Games, the estimated times given would have been considered almost unbelievable. However, in light of the current 200 m WR (at the time of writing), the times are not so far fetched. The 19 s barrier is on the verge of being broken for $\lambda^2 = 0.50$, while for higher λ^2, the current WR is approached. It is interesting to note that, for $\lambda^2 = 1.00$, the model predicts a time of 19.30 s, quite close to Michael Johnson's 19.32 s. These predictions are ideally for zero-wind readings, while the 19.32 s was assisted with a wind of +0.4 m/s. Barring serious injury, It is possible that

Johnson will again lower his 200 m WR mark this coming summer (1997), so we could very well see times in the range predicted in Table 5.

As a comparison to Keller's prediction of 19.25 s [1], which can be considered a straight-track 200 m ($\lambda^2 = 0$), this model yields $t_{200} = 18.54 + 0.16 = 18.70$ s, with a split of 9.67 s (which is just the prediction for the 100 m WR).

7.2 Indoor 200 m

Indoor tracks have much shorter radii of curvature than do outdoor tracks. The centrifugal forces acting on a sprinter will be much higher for large v_c, so it makes sense that the value of λ assigned to subsequent calculations should be lower than for outdoor ones. This is physically realized by banked turns on indoor tracks, which are generally 2 to 4 feet at maximum height. How much lower a value of λ one should choose probably depends on the height of the particular bank, so again no accurate estimate can be made. Due to the R^1 force dependence, then a λ (λ^2) ratio in the range of 2:1 (4:1) might be expected for an outdoor:indoor ratio (under the assumption that the average maximal velocity about the curve is the same). Accurate time and velocity measurements at the end of each race segment (curves and straights) have been calculated, and accurate measurement of these quantities can help determine validity of the model (see Table 11).

Frank Fredericks of Namibia broke the 20 s barrier indoors in 1996 (see Table 1), setting a new 200 m indoor WR of 19.92 s. This can be used to estimate possible values of λ that could be used. Clearly, any value under $\lambda^2 = 0.60$ is quite reasonable, and in fact the 19.51 s prediction for $\lambda^2 = 0.40$ is attractive, as it does not seem beyond the realm of possibility. This does not follow the 4:1 ratio outlined above, however there is no real reason to believe that it should. The only real stipulation is that indoor values of λ should be smaller than outdoor ones.

7.3 Can the 19 s barrier be broken?

Suppose that a value of $\lambda^2 = 0.60$ holds for outdoor performances (this assumption is based on results of Section 9.1). The predicted 200 m record is 19.08 s, assuming a reaction of +0.16 s (Table 5). The minimum possible time allowed without a false start being called would be 19.02 s (this, of course, assumes no wind speed, for which the predictions have been made; if there is a sufficient legal tail wind, the mark would certainly fall). How should this athlete train in order to break the 19 s barrier?

A 0.4% increase in the value of f would give a raw time of 18.85 s, with a 100 m split of 9.94 s ($v_{100} = 11.10$ m/s). Whereas, a larger decrease of 9% in c (greater "endurance") would yield a raw time of 18.83 s, with a marginally slower split of $t_{100} = 9.95$ s, but a slightly faster $v_{100} = 11.11$ m/s. This is an extreme case, but does show how the model parameters might be useful to athletes and coaches as a training gauge.

Various articles [9, 10] have made attempts to predict the future trends of WR performances, and the former states that a sub-19.0 s 200 m could be realized by 2040 (although it also predicts a 100 m time of 9.49 s to match). The authors of [10] are more optimistic, predicting a WR of 18.97 s being set as early as 2004. While their prediction of 19.52 s for 1977 is off, it might be retroactively made consistent by Michael Johnson's 19.32 s WR from the 1996 Olympic Games. If the predicted times of Table 5 are near accurate, and considering the simple argument above, then the 2004 projection may not be far off the mark.

8 Is the 300 m Now a Short Sprint?

Keller determined that the maximum distance over which an athlete could run using the strategy $f(t) = f$ was $d_{crit} = 291$ m [1]. Likewise, physiologists have suggested that a human cannot run at full speed for longer that 30 s (see [6] and references therein). While the latter study is just over 10 years old, one wonders whether or not d_{crit} has dropped. Alternatively, if a sprinter can run a sub-30 s 300 m, would this entail that the different strategy used for races longer than 291 m no longer applies?

As with the 200 m, Table 7 outlines possible 300 m record times, as run in lane 4 of a standard indoor track. Since the actual (if there is one) value of λ^2 is unknown, a range of 0.30 to 0.50 is chosen in light of the 200 m results. In the case of lane 4, the race is made up of the segments (in meters)

$$\begin{aligned} d &= d_{c1} + d_s + d_{c2} + d_s + d_{c3} + d_s \\ &= 31.16 + 50 + 59.42 + 50 + 59.42 + 50 \ . \end{aligned} \tag{16}$$

The estimated time for the first two choices of λ^2 are under the 30 s barrier by more than half a second, while $\lambda^2 = 0.5$ yields a value of 29.72 s (with reaction). Comparison to the current WR of 32.19 s (Table 1) give time differentials of approximately 2.49 to 3.06 s! The 300 m times may be a product of a decaying fit to the data. However, the time differentials cited appear far too large to be manifestations of statistical error alone, which would suggest that there is an additional mechanism (perhaps physiological in origin) at work over this distance. This approach would suggest that the 300 m is still not a sprint, by the definition of Keller [1].

9 A Practical Application: Donovan Bailey

On Saturday, July 27th, 21:00 EST, Donovan Bailey (DB) of Canada crossed the 100 m finish line in a new WR time of 9.84 s (+0.7 m/s wind). Thanks to excellent documentation of data from this race, it is possible to find an "exact" solution[2] to the equations 6, and hence solve them for the parameters (f, τ, c). The relevant data are given in [8],

$$v_{max} = 12.1 \text{ m/s} , \quad (17)$$

$$d_{v_{max}} = 59.50 \text{ m} , \quad (18)$$

$$v_{100} = 11.5 \text{ m/s} , \quad (19)$$

Since the system equations used are different than Keller's, the maximum velocity will not be simply $v_{max} = f\tau$. The maximum value of $v(t)$ is found to be

$$v_{max} = f\tau + c\tau^2 \ln \left\{ \frac{c}{f/\tau + c} \right\} . \quad (20)$$

wth $dv(t_{max})/dt = 0$. The values $(f, \tau, c) = (7.96, 1.72, 0.156)$ are thus obtained. These can be compared with those obtained in [2] by a least-square fit to the official splits listed in Table 8: $(f, \tau, c) = (6.41, 2.39, 0.20)$. Note that the higher value of f and lower values of τ, c are likely a manifestation of solution method and accounting for reaction time.

9.1 Predicting DB's 200 m times

Using the parameters obtained in Section 9, and the model framework established in Section 5, 200 m times will be obtained for DB as run on both

[2] It is emphasized that, while it is possible to obtain an exact set of values for the parameters, these are not DB's parameters, since the model does not account for wind assistance and drag.

indoor and outdoor tracks. Resulting split times are "raw" (*i.e.* without reaction time), but the final time will be given both with and without reaction time (roughly 0.15 s, which is faster than his 1996 Olympic 100 m final reaction time of 0.174 s).

Table 9 shows calculated times and velocities for DB running in lane 4 ($p = 4$) for varying values of λ^2. Since the actual value of this parameter is unknown, in order to determine its possible value predicted times will be matched with DB's past 200 m performances. While no conclusive value of λ^2 could be determined from Section 6, perhaps DB's performances can help shed light. The IAAF lists [4] his best 200 m clocking as 20.76 s, with a 20.39 s wind-assisted performance, in 1994. Assuming that his time will be lower in 1997 (but most likely not world class, or sub-20s, due to his training as a 100 m specialist), it wil be assumed that DB is currently capable of running roughly 20.20 to 20.30 s. This would tend to favor a value of λ^2 between 0.50 and 0.70. Predicted indoor performances are listed in Table 10.

For indoor 200 m, Bailey's performance seems to greatly suffer for large λ, which further supports the claim of smaller values for indoor tracks. A 200 m clocking above 21 s is hardly expected by a world class sprinter! In fact, even the 21 s times ($\lambda^2 = 0.40, 0.50$) seem somewhat slow for the 100 m WR holder. These could suggest that the indoor λ be quite low ($\lambda^2 < 0.4$).

Analogous to Table 11, segment times and velocities for DB have been calculated, and are listed in Table 12.

10 Discussion and General Conclusions

This model is not intended to serve as gospel of how sprinters perform; surely, it is crude at best. However, it can be used as a simple tool to gauge what kind of records might be expected, based on present performances. Due to the "loose" statistical fit of the data from lack of points, the WRs of Section 7 may be somewhat overestimated. DB's predicted performances of Section 9.1 are probably more representative of the possible range of λ^2 values that one might realistically expect, if such a model holds. That is, if he is capable of running the 200 m in the range of 20.15 to 20.40 s, then if λ^2 is the same for all runners, possible values lie between $\lambda^2 = 0.40$ and 0.60. A value of $\lambda^2 = 1.00$, while closely reproducing the current 200 m WR, is definitely wrong from this observational point of view: it would greatly underestimate Bailey's potential ($t > 20.60$ s would hardly be expected by a WR holding sprinter, regardless of specialization).

The following points should be considered, though:

- λ^2 is not the same for indoor and outdoor races; indoor tracks would favor lower λ, so long as they are banked
- λ^2 may not be the same for all athletes; 200 m specialists handle turns with greater ease than 100 m specialists. It may be an individual parameter, like (f, τ, c).
- due to physiological considerations (different posture assumed or muscles/joints used, *etc...*), it seems more likely that the values of τ and/or c may change around the curve
- if the effect is purely physical, then the individual lane records should be strictly decreasing from lane 1 to lane 8. The recorded records (Table 3) suggest that a minimal race time is achieved around lane 3 or 4, contributing to the physiological nature of curve running.

The results of this paper are limited by the availability of relevant data, unfortunately. It would perhaps be of future interest to investigate the physical nature of the parameters (f, τ, c, λ) through study of various athletes. By knowing the effects of their variability on predicted times, models such as these could perhaps be used as a new training tool to gauge and direct the training of World Class athletes.

11 Appendix

Since the initial writing of this paper, there have been several events which can be used to verify the validity of the curve model. The most useful of these took place at Skydome in Toronto on June 1st, the much-hyped "One-to-One Challenge of Champions" which pitted Bailey against 200 m WR holder and Olympic Champion Michael Johnson, in a vein attempt to decide who was the "World's Fastest Man". Bailey won the extremely controversial race over an unconventional distance of 150 m (the middle-ground between each sprinter's specialty), while opponent Johnson pulled up at the 100 m mark with a strained quadriceps. The winning time was 14.99 s, a mere 0.02 s off the "official" World Record of 14.97 s, held by Britain's Linford Christie (the 150 m is not recognized as a standard event, and thus cannot be assigned an official world record [4]).

A number of predictions were made in the buildup to this race as to the value of the winning time. Even contests were held, with prizes going to the individual who could correctly guess the victor and his corresponding victorious time. The majority of these predictions were well under the official 14.99 s finish (see *e.g.* [2, 12, 13]), and most fell in the range of 14.70 − 14.80 s. So, to add to the disappointment of Johnson's dropping out mid-way due to injury was Bailey's curiously "slow" victory.

Thanks to well-documented times for the race, splits were obtained for the 50 m and 100 m marks; the former is on the curve, the latter not. The official splits [14] for Donovan Bailey are given in Table 13. The configuration of the track used in Skydome was $d_c = d_s =$ 75 m (herafter denoted as 75m+75m), and was to have had a radius of curvature corresponding to lanes 8 and 9 of a standard outdoor track [15].

The event itself was plagued with financial, administrative, and organizational problems. The night prior to competition, Bailey threatened to drop out of the race, citing that the track did not conform to the specifications agreed upon in a previously signed contract. In particular, he claimed that the curvature of his lane corresponded more to lane 3 of an outdoor track rather than lane 8. Additionally, he submitted that the curve was 10 m longer than anticipated, giving a 85m+65m configuration instead of 75m+75m [16].

To better match the split times for straightforward comparison, all calculated times include reaction time $t_{reac} = +0.171$ s. Tables 14, 15 give the splits for a 75m+75m configuration as run by Bailey in lanes 3 and 8, respectively. Tables 16, 17 present the same information for an 85m+65m configuration. Note that in this case, the splits would be equal up to 75m, since (for equal lane assignments) the curve is of the same radius despite the fact that it is longer. So, splits are only given for 85m and beyond. The sixth column lists the sum of difference of squares (a loose measure of relative error from a small sample space) $\Sigma^2 \equiv \sum_{i=50,100,150} \Delta_i^2/T_i^2$, where $\Delta_i = t_i - T_i$, for T_i the official splits of Table 13 and t_i the associated model predictions.

The earlier findings in Section 9.1 showed that λ^2 could realistically assume a value between 0.50 and 0.80, so similar values are used here. For interest's sake, possible 200 m times for Bailey are extrapolated, to gauge whether or not he could be a viable contender to Michael Johnson in the 200 m, as stated in his post-race interview.

Upon first inspection, the model reproduces the official splits and race time surprisingly well. The smallest value of Σ^2 for each configuration is taken to be the closest fit to the official race splits. These are:

- 75m+75m, lane 3: $\lambda^2 = 0.50; \Sigma^2 = 8.48 \times 10^{-5}$
- 75m+75m, lane 8: $\lambda^2 = 0.80; \Sigma^2 = 7.90 \times 10^{-5}$
- 85m+65m, lane 3: $\lambda^2 = 0.50; \Sigma^2 = 2.71 \times 10^{-5}$
- 85m+65m, lane 8: $\lambda^2 = 0.70; \Sigma^2 = 4.22 \times 10^{-5}$

Interestingly enough, the closest match above comes from the 85m+65m lane 3 configuration, which is the configuration allegedly used contrary to the signed contracts. The different solutions arise from the readjustment of the ratio λ/R; hence, to narrow down an "exact" solution (if indeed one exists), one needs to analyze other race splits. An interesting test of the model will come at the end of June 1997, when Donovan Bailey will run a 150 m race at a Grand Prix meet in Sheffield, England (most likely on a 50 m+100 m configuration; see [12] for Bailey's possible 150 m times on such a track)[3]

Early this season (1997), Bailey clocked a 20.65 s 200 m [17], and more recently a wind-aided 20.14 s in Oslo. Although injured at the time of the

[3] Bailey won the 150 m in Sheffield, but due to inclement weather conditions (cold rain, running into a headwind), his time was only 15.01 s. Under favorable conditions, he may have been able to run up to 0.2 s faster, but this conjecture is based on experience only. Using the wind-correction method discussed in [7], one can roughly gauge his time to be around 14.87 s in still-air conditions, which would still be slower than an ideal race due colder weather conditions.

latter race, it is quite conceivable that the strong tail-wind could compensate and push his race to an effective peak performance. This time closely agrees with the cited range of λ^2. Assuming that the same value of λ^2 held for the 150 m race in Skydome (between 0.7–0.8), then according to Table 15, a possible configuration of the track would have been 75m+75m with a curve radius equivalent to lane 8 (5.78 s, 10.27 s, 14.92 s). Accounting for the fact that Bailey was undoubtedly much more mentally and physically prepared for the Skydome match, then it is likely that his equivalent outdoor 200 m time would drop. Both Frank Fredericks of Namibia and Ato Boldon of Trinidad are comparable 100 m runners, and each has clocked a sub–20 s 200 m time (19.68 s (1996 Olympic Games) and 19.77 s (1997, Stuttgart Grand Prix), respectively [4]). In fact, Boldon recorded the fastest-ever one-day 100 m/200 m double at Stuttgart (9.90 s/19.77 s). Undoubtedly, it is high muscular endurance which allows them to do this.

For each case listed above, Bailey's extrapolated 200 m times of Tables 14-17 are generally less than for a standard outdoor track. This is simply due to the fact that the longer curve outdoors (100 m v.s. 75 m) creates a larger drain on $f(t)$. Whether or not such seemingly large time discrepancies are physically realizable, or are just a manifestation of the model, are unknown. The proof is left to the sprinter.

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[16] Toronto Slam! Sports web site, http://www.canoe.ca/BaileyJohnson/jun1_protest.html; a public statement of Bailey's protests (01 June 1997)

[17] Official 1997 Outdoor Canadian Rankings, Ontario Track and Field Association web site, http://www.lglobal.com/~ontrack/

Table 1: Men's Sprint World Records as of March 1997. Wind speed of 'i' indicates indoor performance; 'A' indicates performance at altitude.

Event	$t(s)$	v_w (m/s)	Athlete	Location	Date
50 m	5.56	i	Donovan Bailey (CAN)	Reno, NV	9 Feb 1996
55 m	5.99	i, A	Obadele Thompson (BAR)	Colorado Springs, CO	22 Feb 1997
60 m	6.41	i	Andre Cason (USA)	Madrid, ESP	14 Feb 1992
100 m	9.84	+0.7	Donovan Bailey (CAN)	Atlanta, GA	27 Jul 1996
200 m	19.92	i	Frank Fredericks (NAM)	Liévin, FR	18 Feb 1996
	19.32	+0.4	Michael Johnson (USA)	Atlanta, GA	1 Aug 1996
300 m	32.19	i	Robson daSilva (BRA)	Karlsruhe	24 Feb 1989
400 m	44.63	i	Michael Johnson (USA)	Atlanta, GA	4 Mar 1995
	43.29		Harry Reynolds (USA)	Zurich	17 Aug 1988

Table 2: Model predictions of Men's Sprint WRs; $t_{raw} = t_{race} - t_{reac}$, where $t_{reac} = 0.16$ s for all races except 100 m (where it has a known value of 0.17 s).

Event	t_{race}	t_{raw}	t_{fit} (Keller)	t_{fit} (Tibs.-Keller)
50 m	5.56	5.40	5.40	5.40
55 m	5.99	5.83	5.83	5.83
60 m	6.41	6.25	6.26	6.25
100 m	9.84	9.67	9.67	9.67

Table 3: World records by lane for 200 m (from [11]).

Lane	Athlete	t_{200}	Location	Date
1	John Carlos USA	20.12A	Mexico City	16 Oct 68
	Daniel Effiong NIG	20.15	Zurich	04 Aug 93
2	Robson da Silva BRA	20.00	Barcelona	10 Sep 89
3	Michael Johnson USA	19.32	Atlanta	01 Aug 96
4	Pietro Mennea ITA	19.72A	Mexico City	12 Sep 79
	Michael Johnson USA	19.79	Goteborg	11 Aug 95
5	Michael Johnson USA	19.66	Atlanta	23 Jun 96
6	Joe DeLoach USA	19.75	Seoul	28 Sep 88
7	Carl Lewis USA	19.80	Los Angeles	08 Aug 84
8	Michael Johnson USA	19.79	New Orleans	28 Jun 92

Table 4: Keller parameter ($f = 10.230, \tau = 1.147$) predicted outdoor 200 m World Records for various values of λ^2, assuming race is run in lane 4. v_{100} is the velocity for the given split.

λ^2	v_{100}	t_{100}	t_{200}	$t_{200} + 0.16$
0.50	11.36	9.88	18.44	18.60
0.60	11.29	9.92	18.49	18.65
0.70	11.23	9.95	19.52	18.68
0.80	11.17	9.99	18.57	18.73

Table 5: TK parameter ($f = 9.596, \tau = 1.274, c = 0.058$) predicted outdoor 200 m World Records for various values of λ, assuming race is run in lane 4. v_{100} is the velocity for the given split.

λ^2	v_{100}	t_{100}	t_{200}	$t_{200} + 0.16$
0.50	11.14	9.92	18.86	19.02
0.60	11.06	9.97	18.92	19.08
0.70	10.98	10.02	18.98	19.14
0.80	10.91	10.06	19.03	19.19

Table 6: Predicted indoor 200 m World Records for various values of λ, assuming race is run in lane 4.

λ^2	t_{50}	t_{100}	t_{150}	t_{200}	$t_{200} + 0.16$
0.20	5.50	9.82	14.40	18.98	19.14
0.30	5.55	9.88	14.56	19.17	19.33
0.40	5.60	9.95	14.72	19.35	19.51
0.50	5.64	10.01	14.86	19.52	19.68
0.60	5.69	10.08	15.00	19.68	19.84

Table 7: Predicted indoor 300 m World Records, as run in lane 4.

λ^2	t_{50}	t_{100}	t_{150}	t_{200}	t_{250}	t_{300}	$t_{300} + 0.16$
0.30	5.52	9.87	14.55	19.13	24.08	28.99	29.15
0.40	5.55	9.93	14.69	19.29	24.33	29.27	29.43
0.50	5.59	10.00	14.85	19.47	24.59	29.56	29.72

Table 8: Predicted splits (s) and speed (m/s) compared with official for Bailey's 100 m final in Atlanta. Reaction time is rounded to +0.17 s.

Split	10 m	20 m	30 m	40 m	50 m	60 m	70 m	80 m	90 m	100 m
Speed (s)	9.32	10.95	11.67	11.99	12.10	12.10	11.99	11.85	11.67	11.47
Raw	1.89	2.90	3.79	4.64	5.47	6.29	7.12	7.96	8.81	9.67
+reaction	2.06	3.07	3.96	4.81	5.64	6.46	7.29	8.13	8.98	9.84
Official	1.9	3.1	4.1	4.9	5.6	6.5	7.2	8.1	9.0	9.84

Table 9: Bailey's predicted outdoor 200 m times, as run in lane 4.

λ^2	t_{50}	v_{50}	t_{100}	v_{100}	t_{150}	t_{200}	$t_{200} + 0.16$
0.25	5.53	11.74	9.89	11.03	14.56	19.81	19.97
0.36	5.55	11.60	9.98	10.85	14.69	19.96	20.12
0.50	5.59	11.43	10.09	10.65	14.84	20.13	20.29
0.60	5.61	11.31	10.16	10.51	14.93	20.24	20.40
0.70	5.63	11.20	10.24	10.39	15.09	20.43	20.59

λ^2	t_{50}	t_{100}	t_{150}	t_{200}	$t_{200} + 0.16$
0.20	5.62	9.91	14.88	20.32	20.48
0.30	5.68	10.01	15.17	20.71	20.87
0.40	5.75	10.13	15.43	21.05	21.21
0.50	5.81	10.22	15.67	21.37	21.53
0.60	5.88	10.32	15.91	21.68	21.84
0.70	5.94	10.42	16.13	21.97	22.13
0.80	5.99	10.50	16.33	22.23	22.39

Table 10: Bailey's predicted indoor 200 m times, as run in lane 4.

Table 11: TK parameter times and velocities for curve ($c1 = 40.58$ m, $c2 = 59.42$ m), and straight ($s1 = 50$ m) race segments for indoor 200 m.

λ^2	t_{c1}	v	t_{s1}	v	t_{c2}	v
0.20	4.67	11.16	8.99	11.63	14.40	10.70
0.30	4.71	10.95	9.05	11.62	14.56	10.47
0.40	4.75	10.76	9.11	11.61	14.72	10.27
0.50	4.78	10.59	9.16	11.60	14.86	10.09
0.60	4.82	10.43	9.22	11.59	15.00	9.92

Table 12: Bailey parameter times and velocities for curve ($c1 = 40.58$ m, $c2 = 59.42$ m), and straight ($s1 = 50$ m) race segments for indoor 200 m.

λ^2	t_{c1}	v	t_{s1}	v	t_{c2}	v
0.20	4.79	11.20	9.07	11.58	14.88	9.40
0.30	4.84	10.90	9.17	11.53	15.17	9.06
0.40	4.89	10.62	9.26	11.49	15.43	8.80
0.50	4.94	10.38	9.34	11.46	15.67	8.56
0.60	4.99	10.15	9.43	11.42	15.91	8.35

Distance (m)	Split (s)
0	0.171
50	5.74
100	10.24
150	14.99

Table 13: Donovan Bailey's official splits for the Challenge of Champions 150m race at Skydome, Toronto, 01 June 1997 [14].

λ^2	t_{50}	t_{75}	t_{100}	t_{150}	Σ^2	t_{200}
0.50	5.76	7.98	10.21	14.87	8.48×10^{-5}	20.13
0.60	5.79	8.03	10.28	14.94	1.02×10^{-4}	20.22
0.70	5.81	8.09	10.36	15.04	2.97×10^{-4}	20.33
0.80	5.84	8.13	10.41	15.10	6.34×10^{-4}	20.41

Table 14: Bailey's predicted splits as run in lane 3 for a 75m+75m track configuration. All times include reaction time $t_{react} = +0.17$s.

λ^2	t_{50}	t_{85}	t_{100}	t_{150}	Σ^2	t_{200}
0.50	5.73	7.91	10.12	14.75	3.97×10^{-4}	20.00
0.60	5.75	7.95	10.17	14.82	1.79×10^{-4}	20.07
0.70	5.77	7.99	10.22	14.88	8.49×10^{-5}	20.15
0.80	5.78	8.02	10.27	14.92	7.90×10^{-5}	20.20

Table 15: Predicted splits as run in lane 8 for a 75m+75m track configuration.

λ^2	t_{50}	t_{85}	t_{100}	t_{150}	Σ^2	t_{200}
0.50	5.76	8.89	10.26	14.94	2.71×10^{-5}	20.22
0.60	5.79	8.96	10.34	15.04	1.82×10^{-4}	20.33
0.70	5.81	9.02	10.41	15.13	5.12×10^{-4}	20.44
0.80	5.84	9.08	10.48	15.22	1.09×10^{-3}	20.48

Table 16: Predicted splits as run in lane 3 for an 85m+65m track configuration.

λ^2	t_{50}	t_{85}	t_{100}	t_{150}	Σ^2	t_{200}
0.50	5.73	8.80	10.15	14.80	2.41×10^{-4}	20.06
0.60	5.75	8.84	10.20	14.87	8.24×10^{-5}	20.13
0.70	5.77	8.89	10.26	14.94	4.22×10^{-5}	20.22
0.80	5.78	8.93	10.30	15.00	8.34×10^{-5}	20.29

Table 17: Predicted splits as run in lane 8 for an 85m+65m track configuration.