Assignment #5
Physics 322
Due: Monday 20 April 2010

1. (a) Show that two-qubits initially in the product state $|a\rangle b$ (where $a, b = 0, 1$) become entangled by first applying the Hadamard gate to the first qubit, then the CNOT gate applied to the entire state.
(b) Draw the corresponding quantum circuit diagram.
(c) Derive the matrix form of this operation, and show that it produces the correct output for all states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

2. For the problems we’ve discussed so far involving two spin-1/2 systems, we have made use of the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The great utility of Quantum Computing and Quantum Information Theory is that these rely on entanglement of two (or more) quantum states, as we discussed in class. So, frequently it makes more sense to build states (quantum registers) out of the entangled basis $\{\phi^+, \psi^+, \phi^-, \psi^-\}$, where we define $|\phi^\pm\rangle = \frac{1}{\sqrt{2}} [|00\rangle \pm |11\rangle]$, and $|\psi^\pm\rangle = \frac{1}{\sqrt{2}} [|01\rangle \pm |10\rangle]$.

Verify that the entangled basis is in fact a complete basis which spans a 2-D Hilbert Space ($i.e.$ prove orthogonality of each element as well as the closure relation), and is in fact produced by your circuit in question 1.

3. Consider the following quantum circuit (an “inverted” CNOT gate):

(a) Determine the matrix form of this gate from the combination of unitary gates on the “left-hand-side” of the diagram.

(b) For the “right-hand-side” of the diagram equation, label the input qubits $|a\rangle$ and $|b\rangle$ and complete the “logic labels” on the output qubits. Also, make a “truth-table” for all possible input states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and their corresponding output.

(c) Apply your matrix to the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, and verify that it does what it is supposed to do! ($i.e.$ verify your results from part (b)).
4. Beam me up, Scotty! The following question deals with *Quantum Teleportation*, a phenomenon which is true to its name. Suppose two individuals (for security purposes, we’ll call them A and B!) perform an entanglement experiment on two spin states. The two entangled particles can be expressed in the entangled basis described in Question 3. Again for security reasons, A and B take their respective particles and separate (they put it in a box, say, so as not to make a measurement on whatever it is).

Some time passes, and A decides to send to B a particle in the state $|\xi_C\rangle = a|0\rangle + b|1\rangle$. To do so, A takes his superposed state and couples it with $|\xi_C\rangle$. Suppose A’s particle is $|\phi^+\rangle_{AB}$, with the subscript indicating that the particle is entangled with that of B. This implies that the coupled state is, say, $|\Psi_{CAB}\rangle = |\xi_C\rangle_C|\phi^+\rangle_{AB}$.

(a) Show that

$$|\Psi_{CAB}\rangle = \alpha |\phi^+\rangle_{CA} |\xi_B\rangle + \beta |\psi^+\rangle_{CA} \sigma_x |\xi_B\rangle + \gamma |\psi^-\rangle_{CA} (-i\sigma_y) |\xi_B\rangle + \delta |\phi^-\rangle_{CA} \sigma_z |\xi_B\rangle$$

where the $\sigma_k$ are the usual Pauli matrices [Hint: express everything as combinations of $|0\rangle, |1\rangle$. Then note things like $|00\rangle = |\phi^+\rangle + |\phi^-\rangle$].

(b) What are $\alpha, \beta, \gamma, \delta$? What does this tell you about the outcomes of the measurement on the entangled basis for $|\Psi\rangle$? This is called a Bell measurement (when you project onto the entangled basis).

This may just look like some messy algebraic manipulation, but examine closely what the result of (a) says. The state $|\xi_C\rangle_{C} |\Psi\rangle_{AB}$ has been transformed to a new state $|\Phi\rangle_{CA} |\xi_B\rangle$ which can be collapsed by a suitable measurement. The upshot of this process is that A makes a Bell Measurement to determine the state of his particle, and sends a communiqué to B (classical information) informing him/her of the result. B then makes the appropriate Bell measurement on his system, and collapses the state ... to $|\xi_B\rangle$! That is, by performing the measurement B has *recreated* the particle $|\xi_C\rangle$, wherever he may be. Note that the particle really has “teleported” from A’s location to B’s, because when A performs the measurement to determine which entangled state to measure, $|\xi_C\rangle$ is destroyed (this is called the “No cloning” theorem – you cannot duplicate the particle in this manner).