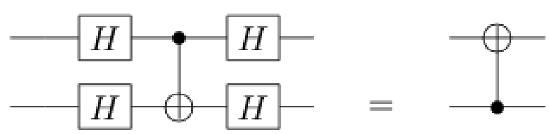
Assignment #5
Physics 322
Due: Monday 13 April 2009

1. In class we discussed the Deutsch algorithm, a code designed to demonstrate whether the function $f : \{0,1\} \longrightarrow \{0,1\}$ is constant or balanced. The Deutsch-Josza algorithm is the generalized extension for an n-qubit input, $f : \{0,1\}^n \longrightarrow \{0,1\}$.

   Consider the D-J algorithm for a 3-qubit input with input state $|\psi\rangle_{\text{input}} = |0\rangle|0\rangle|1\rangle$. Work through the algorithm and show that the function is constant with $f(x) = 1$ if (and only if) the measured output on the first two qubits is $|00\rangle$

2. Consider the following quantum circuit (an "inverted" CNOT gate):



   (a) Determine the matrix form of this gate from the combination of unitary gates on the "left-hand-side" of the diagram.

   (b) For the "right-hand-side" of the diagram equation, label the input qubits $|a\rangle$ and $|b\rangle$ and complete the "logic labels" on the output qubits. Also, make a "truth-table" for all possible input states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and their corresponding output.

   (c) Apply your matrix to the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, and verify that it does what it is supposed to do! (*i.e.* verify your results from part (b)).

3. For the problems we've discussed so far involving two spin-1/2 systems, we have made use of the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The great utility of Quantum Computing and Quantum Information Theory is that these rely on entanglement of two (or more) quantum states, as we discussed in class. So, frequently it makes more sense to build states (quantum registers) out of the *entangled basis* $\{|\phi^+\rangle, |\psi^+\rangle, |\phi^-\rangle, |\psi^-\rangle\}$, where we define $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}[|00\rangle \pm |11\rangle]$, and $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}[|01\rangle \pm |10\rangle]$.

   (a) Verify that the entangled basis is in fact a complete basis which spans a 2-D Hilbert Space (*i.e.* prove orthogonality of each element as well as the closure relation).

(b) Using the one-qubit Hadamard gate and the two-qubit CNOT gate, show that the entangled basis can be constructed from the usual two-state basis [Hint: remember what each gate does, and apply them in sequence!]

(c) Draw quantum circuits for the basis transformations you derived in part (b).

4. Beam me up, Scotty! The following question deals with *Quantum Teleportation*, a phenomenon which is true to its name. Suppose two individuals (for security purposes, we'll call them A and B!) perform an entanglement experiment on two spin states. The two entangled particles can be expressed in the entangled basis described in Question 3. Again for security reasons, A and B take their respective particles and separate (they put it in a box, say, so as not to make a measurement on whatever it is).

Some time passes, and A decides to send to B a particle in the state $|\xi\rangle_C = a|0\rangle + b|1\rangle$. To do so, A takes his superposed state and couples it with $|\xi\rangle_C$. Suppose A's particle is $|\phi^+\rangle_{AB}$, with the subscript indicating that the particle is entangled with that of B. This implies that the coupled state is, say, $|\Psi\rangle_{CAB} = |\xi\rangle_C\, |\phi^+\rangle_{AB}$.

(a) Show that

$$\begin{aligned} |\Psi\rangle_{CAB} &= \alpha\, |\phi^+\rangle_{CA}|\xi\rangle_B + \beta\, |\psi^+\rangle_{CA}\, \sigma_x|\xi\rangle_B \\ &\quad + \gamma\, |\psi^-\rangle_{CA}\, (-i\sigma_y)\, |\xi\rangle_B + \delta\, |\phi^-\rangle_{CA}\, \sigma_z\, |\xi\rangle_B \end{aligned}$$

where the σ_k are the usual Pauli matrices [Hint: exrpess everything as combinations of $|0\rangle, |1\rangle$. Then note things like $|00\rangle = |\phi^+\rangle + |\phi^-\rangle$].

(b) What are $\alpha, \beta, \gamma, \delta$? What does this tell you about the outcomes of the measurement on the entangled basis for $|\Psi\rangle$? This is called a Bell measurement (when you project onto the entangled basis).

This may just look like some messy algebraic manipulation, but examine closely what the result of (a) says. The state $|\xi\rangle_C\, |\Psi\rangle_{AB}$ has been transformed to a new state $|\Phi\rangle_{CA}\, |\xi\rangle_B$ which can be collapsed by a suitable measurement. The upshot of this process is that A makes a Bell Measurement to determine the state of his particle, and sends a communiqué to B (classical information) informing him/her of the result. B then makes the appropriate Bell measurement on his system, and collapses the state ... to $|\xi\rangle_B$! That is, by performing the measurement B has *recreated* the particle $|\xi\rangle_C$, wherever he may be. Note that the particle really has "teleported" from A's location to B's, because when A performs the measurement to determine which entangled state to measure, $|\xi\rangle_C$ is destroyed (this is called the "No cloning" theorem – you cannot duplicate the particle in this manner).