Assignment #4 solutions

Physics 322

1. (a) If the original state is

$$|\psi(0)\rangle = \frac{1}{2}|++\rangle + \frac{1}{2}|+-\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

then a measurement of the S_{1z} spin which yields $-\hbar/2$ will ultimately represent finding the state $|--\rangle$. This will occur with a probablity of $\boxed{P = |1/\sqrt{2}|^2 = 1/2}$. Afterward, the state has collapsed to $|--\rangle$.

(b) In order to measure S_{1x}, we must re-write the state $--\rangle$ in the basis $\{|+\rangle_x, |-\rangle_x$. Recall that $|-\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x)$, so we get

$$|--\rangle = \frac{1}{2}(|+\rangle_x - |-\rangle_x) \otimes (|+\rangle_x - |-\rangle_x)$$

$$|--\rangle = \frac{1}{2}(|++\rangle_x - |-+\rangle_x - |+-\rangle_x + |--\rangle_x)$$

There are clearly two of the four states in which the first particle is $|-\rangle_x$, for which a measurement of the spin would return $-\hbar/2$. So, the probability is $\boxed{P = \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{2}}$.

Likewise, a measurement of $|+\rangle_x$ (eigenvalue $+\hbar/2$) would result in the same probability, for the same reason.

(c) The state is once again in the original superposition of (a). We are looking for the states $|++\rangle$, or $|--\rangle$. In this case, we will find the two spins of $+\hbar/2$ with a 1/4 probability, and $|--\rangle$ with the 1/2 probability found in (a).

2. If we define $S_a^{(1)} = S_{1z}$, and then $S_b^{(2)} = \cos\theta S_{2z} + \sin\theta S_{2x}$, then we can consider the action of these matrices on the singlet state $|00\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$. The states $\{|+\rangle|-\rangle\}$ are eigenstates of the S_z matrices, but not S_x, which gives $S_x|\pm\rangle = \mp\hbar/2|\mp\rangle$. So, we can write the expectation $\langle S_a^{(1)} S_b^{(2)}\rangle = \langle 00|S_a^{(1)} S_b^{(2)}|00\rangle$ as:

$$\frac{1}{\sqrt{2}}(\langle +-| - \langle -+|) S_{1z}(\cos\theta S_{2z} + \sin\theta S_{2x})\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

We can break this into two pieces: one in which the operator $S_{1z}S_{2z}$ acts, and the other in which the operator $S_{1z}S_{2x}$ acts:

$$\begin{aligned} S_{1z}s_{2z} \quad \longrightarrow \quad & \frac{1}{\sqrt{2}}\cos\theta S_{1z}S_{2z}(|+-\rangle - |-+\rangle) \\ = \quad & \frac{1}{\sqrt{2}}\cos\theta\left(-\frac{\hbar^2}{4}|+-\rangle + \frac{\hbar^2}{4}|-+\rangle\right) \\ = \quad & -\frac{\hbar^2}{4}\cos\theta|00\rangle \end{aligned}$$

and

$$\begin{aligned}
S_{1z}s_{2x} &\longrightarrow \frac{1}{\sqrt{2}}\sin\theta S_{1z}S_{2x}\left(|+-\rangle - |-+\rangle\right) \\
&= \sin\theta\left(-\frac{\hbar^2}{4}|++\rangle + \frac{\hbar^2}{4}|--\rangle\right) \\
&= \frac{1}{\sqrt{2}}\frac{\hbar^2}{4}\sin\theta\left(-|11\rangle + |1,-1\rangle\right)
\end{aligned}$$

Since the second operation results in a mixture of states that are orthogonal to $|00\rangle$, the expectation value can be calculated exclusively from the action of $S_{1z}S_{2z}$, giving $\boxed{\langle S_a^{(1)}S_b^{(2)}\rangle = \langle 00|\cos\theta S_{1z}S_{2z}|00\rangle = -\frac{\hbar^2}{4}\cos\theta}$

③ Define $|00\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$.

$\Rightarrow S_{1z} S_{2z} = S_{1z} \otimes S_{2z}$

$$= \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & 0\\ 0 & -1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{pmatrix}$$

$$\therefore\ S_{1z} S_{2z} = \frac{\hbar^2}{4}\begin{pmatrix}1 & & & \\ & -1 & & 0\\ & & -1 & \\ 0 & & & 1\end{pmatrix}$$

Also,

$$S_{1z} S_{2x} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{pmatrix}$$

Thus $\Rightarrow \langle S_a^1 S_b^2\rangle = \langle 00|\, S_{1z} S_{2z} \cos\theta + S_{1z} S_2 \sin\theta\, |00\rangle$

$$= \frac{1}{2}\frac{\hbar^2}{4}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}\left[\begin{pmatrix}\cos\theta & \sin\theta & & \\ \sin\theta & -\cos\theta & & 0\\ & & -\cos\theta & -\sin\theta\\ & 0 & -\sin\theta & \cos\theta\end{pmatrix}\right]\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$= \frac{\hbar^2}{8}\begin{pmatrix}\sin\theta & -\cos\theta & \cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$= \frac{\hbar^2}{8}\left(-\cos\theta - \cos\theta\right)$$

$$\boxed{\langle S_a^1 S_b^2\rangle = -\frac{\hbar^2}{4}\cos\theta}$$

TADA!

4. The general action of $\mathcal{J}_-$ on the state $|J, M\rangle$ is $\mathcal{J}_-|J, M\rangle = \hbar\sqrt{J(J+1) - M(M-1)}|J, M-1\rangle$. So, we get

$$\mathcal{J}_-|1,1\rangle = \hbar\sqrt{2}|1,0\rangle$$

This implies that

$$|1,0\rangle = \frac{1}{\hbar\sqrt{2}}\mathcal{J}_-|1,1\rangle$$

We can re-write the RHS by replacing $|1,1\rangle = |++\rangle$, and using the identity $\mathcal{J}_- = S_{1-} + S_{2-}$. Also, recall that $S_-|+\rangle = \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}|-\rangle = |-\rangle$, which implies:

$$(S_{1-} + S_{2-})|++\rangle = \frac{\hbar}{\hbar\sqrt{2}}\left(|-+\rangle + |+-\rangle\right)$$

Thus, substituting this back into the previous RHS expression, we find $\boxed{|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)}$, which is the correct expression for the state.

Acting $\mathcal{J}_-$ again on this we obtain the general expression for $|1,-1\rangle$, which is $\mathcal{J}_-|1,0\rangle = \hbar\sqrt{2}|1,-1\rangle$. Proceeding as before, we find:

$$\mathcal{J}_-|1,0\rangle = (S_{1-} + S_{2-})\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

$$\begin{aligned} \implies \quad & \frac{1}{\sqrt{2}}(S_{1-} + S_{2-})((|+-\rangle + |-+\rangle) \\ & \frac{1}{\sqrt{2}}(\hbar|--\rangle + \hbar|--\rangle = \sqrt{2}|--\rangle \end{aligned}$$

Substituting this expression back into the original gives $\boxed{|1,-1\rangle = \frac{1}{\sqrt{2}\hbar}\sqrt{2}\hbar|--\rangle = |--\rangle}$, which is exactly what we should expect! So, the triplet states are well-defined by the lowering operations on the original state $|1,1\rangle = |++\rangle$.

(b) If we assume that the singlet state $|J, M\rangle = |0,0\rangle$ is a linear combination of the states $|+-\rangle$ and $|-+\rangle$, then we have:

1. Normalization:

$$\begin{aligned} \langle 0,0|0,0\rangle = 1 = (\alpha^*\langle +-| + \beta^*\langle -+|)(\alpha|+- \; + \beta|-+\rangle) \\ \implies \quad 1 = |\alpha|^2 + |\beta|^2 \end{aligned}$$

2. Orthogonality:

$$\langle 1,0|00\rangle = 0 = \frac{1}{\sqrt{2}}(\langle +-| + \langle -+|)(\alpha|+-\rangle + \beta|-+\rangle$$
$$\implies \quad 0 = \alpha + \beta$$

and so $\alpha = -\beta$.

Using this constraint in the normalization conditions gives $\alpha = -\beta = \frac{1}{\sqrt{2}}$, thus we recover the singlet state: $\boxed{|0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)}$

$$|0,0\rangle = \alpha|+-\rangle + \beta|-+\rangle$$

Using normalization of $|0,0\rangle$ and orthogonality with $|1,0\rangle$, show that α and β must give the proper singlet combination.