Assignment #4
Physics 322
Due: Wednesday 01 April 2009 (no foolin'!)

Please answer all questions, and turn in your solutions as a hard copy. If you use Maple, you may turn in your Maple output, so long as it is well-documented (not just Maple code and output). All questions are of equal value.

1. Prove that the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ is entangled, in the sense that it cannot be written as a pure state $|\psi\rangle = |\phi\rangle \otimes |\xi\rangle$, where $|\phi\rangle = a|0\rangle + b|1\rangle$ and $|\xi\rangle = c|0\rangle + d|1\rangle$. [Hint: Assume it *can* be written that way, and then show that no unique complex numbers a, b, c, d can exist.]

2. An introduction to the density matrix. For a quantum state $|\psi\rangle$, we define the density matrix ρ as the projection operator $\rho = |\psi\rangle\langle\psi|$. In vector notation, this is the "outer product", in which we're multiplying a column vector by a row vector (the opposite of a dot product). For example:

$$|+\rangle\langle+| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad |+\rangle\langle-| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0\ \ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and so on. Use the same rules as matrix multiplication: a 2×1 vector multiplying a 1×2 vector gives a 2×2 matrix.

(a) For a spin-1/2 state prepared in the general state $|\psi\rangle = \cos\frac{\theta}{2} e^{-i\phi/2}|+\rangle + \sin\frac{\theta}{2} e^{+i\phi/2}|-\rangle$, determine the matrix form of $\rho(\theta, \phi)$ in the S_z basis.

(b) Show that $\rho(\theta, \phi) = [\rho(\theta, \phi)]^2$ for your result in (a). This relation is characteristic of the density matrix for a *pure* state. For example, the state in question is one for a system which has been "polarized" after going through a Stern-Gerlach apparatus, and so it is in an eigenstate of the apparatus.

(c) Consider now a quantum system which has not yet been spin-polarized (*e.g.* before the spin-1/2 system passes through the Stern-Gerlach apparatus). In this case, the spins are aligned (isotropically) in every possible direction. So, the density matrix you derived in part (a) must also be evaluated in every direction. Determine the coefficients of this new matrix $\rho_m(\theta, \phi)$ by averaging the matrix elements over a solid angle:

$$\bar{\rho}_{ij} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \rho_{ij}(\theta, \phi)\ sin\theta\ d\theta\ d\phi$$

(d) The matrix ρ_m you derived in (a) is the density matrix for a *mixed state*, one which represents a statistical mixture of many different states. Does the relationship obtained in (b) still hold?

3. Suppose that we define two multiparticle spin operators as $S_a^{(1)} = S_{1z}$, and $S_b^{(2)} = \cos\theta S_{2z} + \sin\theta S_{2x}$. Evaluate the expectation $\langle S_a^{(1)} S_b^{(2)} \rangle$ for the singlet state, $\psi\rangle = \frac{1}{\sqrt{2}}\left[| +- \rangle - | -+ \rangle \right]$, and show that it is equal to $-\frac{\hbar^2}{4}\cos\theta$. Use only Dirac notation and algebra for this question.

4. Explicitly calculate the result from the last question using the vector form of the singlet state, and the matrix form of the operator $S_{1a}S_{2b}$. Note that you will have to determine the form of each matrix first (hint: tensor product!).