

Assignment #3

Physics 322

Due: Monday 01 March 2010

Please answer all questions with complete solutions. All questions are of equal value.

1. As you are now aware, the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are related to the operators that correspond to the spin components in the x , y , and z directions. These matrices are elements of a matrix group called $SU(2)$. This is the space of all 2×2 matrices that have unit determinant (the ‘S’ in $SU(2)$), and are also *unitary* (the ‘U’ in $SU(2)$).

Show that *any* matrix of the form $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ can be written as a linear combination of the matrices $\{I, \sigma_x, \sigma_y, \sigma_z\}$, where I is the identity matrix. Hence, these 4 matrices can be understood to be a *basis* for all 2×2 matrices [Hint: show that the each coefficients m_{ij} can be written as the linear combination of two of the four basis matrices].

2. The (semi-classical) Bohr model of the hydrogen atom predicted that the energy levels of the hydrogen atom follow the pattern $E_n = -\frac{E_1}{n^2}$, where $E_1 = \frac{e^2}{2a_0} = 13.6$ eV is the ionization energy, and $a_0 = \frac{\hbar^2}{me^2}$ is the Bohr radius.

(a) Calculate the energy expectation value of the 1s, 2s, and 3s states, $\langle H \rangle$ (H is the Hamiltonian), and show that the energy levels are in fact those predicted by Bohr. [Hint: your answer won’t look exactly like the Bohr levels at first, but with a bit of algebra it can be done]. The radial wavefunctions are in your book, p. 154 (Table 4.7).

(b) Now calculate the expectation value of the 2p state and 3p state. Discuss any corrections to the Bohr levels you might notice.

3. Book problem 4.33.

4. A spin-1/2 particle is prepared in the general state

$$|+\rangle_u = \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}} |+\rangle + \sin \frac{\theta}{2} e^{i\frac{\phi}{2}} |-\rangle$$

Show that the expectation values of the spin operators will be

$$\langle S_x \rangle = \frac{\hbar}{2} \sin \theta \cos \phi, \quad \langle S_y \rangle = \frac{\hbar}{2} \sin \theta \sin \phi, \quad \langle S_z \rangle = \frac{\hbar}{2} \cos \theta$$

The result implies that for large numbers of particles (which is what the average means!), the quantum mechanical spins become the components of a classical angular momentum vector with norm $\hbar/2$.