Assignment #2 solutions

Physics 322

1. The Schrödinger equation is

$$\left[-\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2}r+\frac{l(l+1)}{2\mu r^2}-\frac{e^2}{r}\right]\psi_{nlm}(r,\theta,\phi)=E_{nlm}\psi_{nlm}(r,\theta,\phi)$$

where the wavefunctions $\psi_{nlm}(r,\theta,\phi)=R_{nl}(r)Y_{lm}(\theta,\phi)$. Since the SE does not depend on the angular coordinates, this reduces to the radial equation

$$\left[-\frac{\hbar^2}{2\mu r}\frac{d^2}{dr^2}r+\frac{l(l+1)}{2\mu r^2}-\frac{e^2}{r}\right]R_{nl}(r)=E_{nl}R_{nl}(r)$$

Substituting $R_{nl}(r)=\frac{u_{nl}(r)}{r}$ gives the new differential equation

$$\left[-\frac{\hbar^2}{2\mu r}\frac{d^2}{dr^2}+\frac{l(l+1)}{2\mu r^3}-\frac{e^2}{r^2}\right]u_{nl}(r)=E_{nl}\frac{u_{nl}(r)}{r}$$

from which we can cancel an overall factor of $1/r$. This gives

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)}{2\mu r^2}-\frac{e^2}{r}\right]u_{nl}(r)=E_{nl}u_{nl}(r)$$

In the limit $r\to\infty$, the $1/r$ and $1/r^2$ terms vanish, leaving only the derivative term,

$\boxed{-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_{nl}(r)=E_{nl}u_{nl}(r)}$

which is the free-paricle Schrödinger equation we all know and love!

2. (a) The Laplacian in cylindrical coordinates is

$$\nabla^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}$$

We already know that, in their coordinate representation, the linear and angular momenta in the z-direction are naturally expressed as the conjugate to their respective positions:

$$P_z=\frac{\hbar}{i}\frac{\partial}{\partial z}\quad;\quad;L_z=\frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

So, we can immediately see that the Hamiltonian can be written as

$$H=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}L_z^2+P_z^2+V(\rho)$$

Since only operators of like coordinates will not commute – *e.g.* $[P_z,Z]=0,[L_z,\Phi]=0$ – and the Hamiltonian contains no terms of the form P_ρ, Z, or

Φ (the position operator for the angular coordinate), it becomes immediately clear that $[H, P_z] = 0$ and $[H, L_z] = 0$.

(b) By separation of variables, we assume the solution can be expressed as $\psi(\rho, \phi, z) = f(\rho)g(\phi)h(z)$ (suppressing for the time being the parameter-dependence). Since $\{H, L_z, P_z\}$ form a CSCO, that means that $\psi(\rho, \phi, z)$ is a simultaneous eigenfunction of each operator. Thus:

$$\begin{aligned} P_z\psi(\rho, \phi, z) &= k\hbar\psi(\rho, \phi, z) \\ \frac{\hbar}{i}\frac{\partial}{\partial z}f(\rho)g(\phi)h(z) &= k\hbar f(\rho)g(\phi)h(z) \\ \frac{d}{dz}h(z) &= ikh(z) \\ h(z) &= e^{ikz} \end{aligned}$$

since the PDE becomes an ODE in the z-coordinate, and the other functions cancel out of the equation. By a similar argument using the L_z eigenvalue equation, it follows that $g(\phi) = e^{-i\phi}$. Thus, the CSCO condition implies that $\psi(\rho, \phi, z) = f(\rho)e^{il\phi}e^{ikz}$.

(c) Substituting the function in (b) in lieu of ϕ, we find the Hamiltonian gives a differential equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 f(\rho)}{\partial\rho^2}e^{il\phi}e^{ikz}+\frac{1}{\rho}\frac{\partial f(\rho)}{\partial\rho}e^{il\phi}e^{ikz}-\frac{1}{\hbar^2\rho^2}L_z^2 f(\rho)e^{il\phi}e^{ikz}-\frac{1}{\hbar^2}P_z^2 f(\rho)e^{il\phi}e^{ikz}+V(\rho)f(\rho)e^{il\phi}e^{ikz}$$

$$= Ef(\rho)e^{il\phi}e^{ikz}$$

where we have used the coordinate representation of P_z and L_z to replace the partial derivatives in z and ϕ. Now, the angular momentum term depends on the coordinates x and y, which ultimately determine the coordinates ρ and ϕ. However, the z term has no connection to either, so we can actually remove the P_z dependence from the equation. As we did for the spherically-symmetric Hamiltonian, we can simply replace these terms with their eigenvalue-equivalents to obtain the differential equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial\rho^2}gh + \frac{1}{\rho}\frac{\partial f}{\partial\rho}gh + V(\rho)f(\rho)fgh = E_n fgh$$

The functions f, g cancel out of the equation, and we are left with an ODE in ρ:

$$-\frac{\hbar^2}{2m}\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} - \frac{l^2}{\rho^2}f + V(\rho)f(\rho)f = E_n f$$

This explicitly shows the parameter dependence of $f(\rho)$. In addition to its own internal quantum number n, which defines the orthogonality of the family of functions $f(\rho) \equiv f_n(\rho)$, the dependence is also on l. Hence, the radial function is $\boxed{f_{nl}(\rho)}$, and so the energy eigenvalue depends on n, l. If you didn't argue that it is independent of k, don't worry – that was a subtle distinction!

3. The radial function and totoal wavefunction for tritium in the 1s state is essentially that for hydrogen in the 1s state:

$$\phi_{T,10}(r) = 2a_0^{(-3/2)} e^{-r/a_0} \quad ; \psi_{T,100} = \phi_{T,10} Y_0^0(\theta, \phi)$$

After the decay, the system is still in the 1s state, only now it has a nuclear charge of $Z = +2$:

$$\phi_{He,10}(r) = 2\left(\frac{2}{a_0}\right)^{(3/2)} e^{-2r/a_0} \quad ; \psi_{He,100} = \phi_{He,10} Y_0^0(\theta, \phi)$$

The probability of a decay is thus just the probability that the system starts in the $|T\rangle$ state and ends up in the $|He\rangle$ state:

$$\begin{aligned} P_{T \to He} &= |\,\langle He|T\rangle\,|^2 \\ &= \left| \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \psi^*_{\text{He},100} \psi_{\text{T},100}\; r^2 \sin\theta \; dr \; d\theta \; d\phi \right|^2 \end{aligned}$$

The integral can easily be evaluated with Maple, and we find $\boxed{P_{T \to He} = |0.83|^2 = 0.70}$. Thus, there is a 70% chance of such a decay!

4. Everyone got this one!