

Assignment #2

Physics 322

Due: Wednesday 17 February 2010

Please answer all questions with complete solutions. All questions are of equal value.

1. The Schrödinger equation for the electron in the hydrogen atom is

$$\left[-\frac{\hbar^2}{2\mu r} \frac{\partial^2}{\partial r^2} r + \frac{l(l+1)}{2\mu r^2} - \frac{e^2}{r} \right] \psi_{nlm}(r, \theta, \phi) = E_{nlm} \psi_{nlm}(r, \theta, \phi)$$

where μ is the reduced mass of the system, and $e^2 = \frac{qe q_p}{4\pi\epsilon_0}$. Show that, in the limit $r \rightarrow \infty$, the radial wavefunction behaves like that of a free particle. [Hint: Replace $R_{nl}(r) = \frac{u_{nl}(r)}{r}$ show that the Schrödinger equation becomes that of a free particle with wavefunction $u_{nl}(r)$]. *You don't need Maple to solve this!* Simply do the algebra by hand.

2. In class, we dealt with the angular momentum for a spherically-symmetric system. In many cases, we can consider systems whose symmetry is different. This exercise deals with a particle of mass m in a cylindrically-symmetric potential $V(\rho)$ that is both independent of z and ϕ . Cylindrical coordinates are ρ, ϕ, z , with $x = \rho \cos \phi$, and $y = \rho \sin \phi$.

(a) Starting from the usual expression for the Hamiltonian, $H = -\frac{\hbar^2}{2m} \nabla^2 + V(\rho)$, show that H commutes with P_z and L_z . The Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

and the usual associations are made between operators and their corresponding coordinate-representation differential operators.

It's probably easiest to insert the differential form of the angular momentum operators in the commutator, and make use equality of mixed partial derivatives.

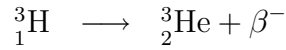
(b) Your answer in (a) shows that $\{H, L_z, P_z\}$ form a CSCO, and thus the complete wavefunction is a simultaneous solution of the eigenvalue equations

$$H|n, l, k\rangle = E|n, l, k\rangle \quad , \quad L_z|n, l, k\rangle = l\hbar|n, l, k\rangle \quad , \quad P_z|n, l, k\rangle = k\hbar|n, l, k\rangle$$

Argue that the eigenfunctions will have the form $\psi_{n,l,k}(\rho, \phi, z) = f(\rho)e^{il\phi}e^{ikz}$.

(c) Just as we did for the spherical potential discussed in class, write down a more explicit differential equation for the ρ -dependent part of the wavefunction, $f(\rho)$. On which quantum numbers (n, k, l) does this function depend?

3. The wavefunction of the hydrogen atom can be used to model quantum mechanical processes of hydrogen-like atoms (*i.e.* atoms with only one electron). For example, we can model the radioactive beta decay of a tritium atom – hydrogen with 2 neutrons – to helium-3 (with one neutron):



Suppose the tritium is in the 1s (ground) state, and has a wavefunction $\psi_{100}(r, \theta, \phi) = R_{1,0}(r)Y_{00}(\theta, \phi)$. After the beta decay, the He-3 atom is still in the 1s state. Determine the decay probability. [Hint: note that the nuclear charge increases from $Z = +1$ to $Z = +2$ during the decay].

Remember that the integration of the radial and spherical harmonic functions is over the volume differential $dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$.

4. Another “Maple-made-it-easy” question! The spin-1/2 operators are defined as $S_j = \frac{\hbar}{2}\sigma_j$, where the $\{\sigma_j\}$ matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1)$$

These are known as the Pauli spin matrices.

(a) Find their eigenvalues and eigenvectors. Are these Hermitian? Are they unitary? (Don’t remember what unitary means for a matrix? Check your notes from last semester!)

(b) Show that the matrices obey similar commutation relations to the angular momentum operators: $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$.