Assignment #2
Physics 322
Due: Wednesday 11 February 2009

Please answer all questions with complete solutions. All questions are of equal value.

1. In class, we dealt with the angular momentum for a spherically-symmetric system. In many cases, we can consider systems whose symmetry is different. This exercise deals with a particle of mass m in a cylindrically-symmetric potential $V(\rho)$ that is both independent of z and ϕ. Cylindrical coordinates are ρ, ϕ, z, with $x = \rho\cos\phi$, and $y = \rho\sin\phi$.

   (a) Starting from the usual expression for the Hamiltonian, $H = -\frac{\hbar^2}{2m}\nabla^2 + V(\rho)$, show that H commutes with P_z and L_z. The Laplacian in cylindrical coordinates is
   $$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$
   and the usual associations are made between operators and their corresponding coordinate-representation differential operators.

   (b) Your answer in (a) shows that $\{H, L_z, P_z\}$ form a CSCO, and thus the complete wavefunction is a simultaneous solution of the eigenvalue equations
   $$H|n,l,k\rangle = E|n,l,k\rangle \quad , \quad L_z|n,l,k\rangle = l\hbar|n,l,k\rangle \quad , \quad P_z|n,l,k\rangle = k\hbar|n,l,k\rangle$$
   Argue that the eigenfunctions will have the form $\psi_{n,l,k}(\rho,\phi,z) = f(\rho)e^{il\phi}e^{ikz}$.

   (c) Just as we did for the spherical potential discussed in class, write down a more explicit differential equation for the ρ-dependent part of the wavefunction, $f(\rho)$. On which quantum numbers (n, k, l) does this function depend?

2. An example of a system that might show such a symmetry is a quantum harmonic oscillator restricted to vibrate in two dimensions. Recall that the raising and lowering operators for a QHO are defined as
   $$a_j = \frac{1}{\sqrt{2}}\left(\beta R_j + \frac{i}{\beta\hbar}P_j\right)$$
   $$a_j^\dagger = \frac{1}{\sqrt{2}}\left(\beta R_j - \frac{i}{\beta\hbar}P_j\right)$$
   where $\beta = \sqrt{\frac{m\omega}{\hbar}}$, and the subscript $j = \{x, y\}$.

   (a) Determine an expression for L_z in terms of the operators $a_x, a_y, a_x^\dagger, a_y^\dagger$.

   (b) Show that $[L_z, H] = 0$, where $H = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1)$ (the Hamiltonian for a 2-D QHO). This means that L_z is a constant of the motion, as indicated before, and furthermore it shows that $\{H, L_z\}$ form a CSCO from which we can determine the system's eigenvectors.

3. The wavefunction of the hydrogen atom can be used to model quantum mechanical processes of hydrogen-like atoms (*i.e.* atoms with only one electron). For example, we can model the radioactive beta decay of a tritium atom – hydrogen with 2 neutrons – to helium-3:

$$^3_1\mathrm{H} \longrightarrow {}^3_2\mathrm{He} + \beta^-$$

Suppose the tritium is in the 1s (ground) state, and has a wavefunction $\psi_{100}(r, \theta, \phi) = R_{1,0}(r)Y_0^0(\theta, \phi)$. After the beta decay, the He-3 atom is still in the 1s state. Determine the decay probability. [Hint: note that the nuclear charge increases from $Z + 1$ to $Z = +2$ during the decay].

Remember that the integration of the radial and spherical harmonic functions is over the volume differential $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

4. Another "Maple-made-it-easy" question! The spin-1/2 operators are defined as $S_j = \hbar/2\sigma_j$, where the $\{\sigma_j\}$ matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{1}$$

These are known as the Pauli spin matrices.

(a) Find their eigenvalues and eigenvectors. Are these Hermitian? Are they unitary?

(b) Show that the matrices obey similar commutation relations to the angular momentum operators: $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$.