1. Since everyone essentially got the first question, I’m not writing up solutions. If you have questions, let me know.

2. Same for the commutation relations – the only issue was that some answers didn’t justify the use of the Levi-Civita symbol, $\epsilon_{ijk}$.

3. (a) In the case of small angles $\phi = \epsilon$, we know that $\sin \epsilon \approx \epsilon$, and $\cos \epsilon \approx 1$. So, the rotation of the vector becomes

$$R_z(\epsilon) = \begin{pmatrix} 1 & \epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \epsilon y \\ -\epsilon x + y \\ z \end{pmatrix}$$

(b) The rotation of the state vector for small angle $\epsilon$ is $R_z \psi(x, y, z) = \psi(x - \epsilon y, \epsilon x + y, z)$, which we can expand as the series

$$\psi(x + \Delta x, y + \Delta y, z) \approx \psi(x, y, z) + \frac{\partial \psi}{\partial x} \Delta x + \frac{\partial \psi}{\partial y} \Delta y + \text{higher order terms}$$

In this case, $\Delta x = -\epsilon y$ and $\Delta y = \epsilon x$, so we find

$$R_z \psi(x, y, z) \approx \psi(x, y, z) - \frac{\partial \psi}{\partial x} y + \frac{\partial \psi}{\partial y} x$$

which we can simplify to

$$R_z \psi(x, y, z) \approx \psi(x, y, z) + \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi(x, y, z)$$

and so

$$R_z \psi(x, y, z) \approx \left( I + x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi(x, y, z)$$

(c) This is just our coordinate representation for the angular momentum operator, so we can thus conclude that $R_z \psi(x, y, z) \approx [1 + \frac{i}{\hbar} L_z]$. In fact, for larger angles, it can be shown that the exact form of the rotation matrix is $R_z = e^{iL_z \hbar}$. The angular momentum operators are said to be the \textit{generators} of the group of rotation matrices. They are actually elements of what’s called a \textit{Lie Algebra}, and the rotation matrices are the corresponding \textit{Lie Group}.

* Note that I made the signs consistent here. In actual fact, the positive or negative sign simply defines the direction of the rotation, and isn’t a big deal.
4. We know that $L_x = \frac{1}{2} (L_+ + L_-)$, and from class we determined that the corresponding matrix form is

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The corresponding eigenvalues and eigenvectors for $L_x$ in terms of those for $L_z$ can thus be pulled directly out of Maple:

$$\phi^x_{+1} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2} \left( \phi_1 + \sqrt{2} \phi_2 + \phi_{-1} \right) \quad \text{Eigenvalue} : \hbar$$

$$\phi^x_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\phi_1 - \phi_{-1}) \quad \text{Eigenvalue} : 0$$

$$\phi^x_{-1} = \frac{1}{2} \begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2} \left( \phi_1 - \sqrt{2} \phi_2 + \phi_{-1} \right) \quad \text{Eigenvalue} : -\hbar$$