Assignment #1 Solutions, Physics 322

1. Since everyone essentially got the first question, I'm not writing up solutions. If you have questions, let me know.

2. Same for the commutation relations – the only issue was that some answers didn't justify the use of the Levi-Civita symbol, ϵ_{ijk}.

3. (a) In the case of small angles $\phi = \epsilon$, we know that $\sin\epsilon \approx \epsilon$, and $\cos\epsilon\, approx 1$. So, the rotation of the vector becomes

$$\mathcal{R}_z(\epsilon) = \begin{pmatrix} 1 & \epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \epsilon y \\ -\epsilon x + y \\ z \end{pmatrix}$$

(b) The rotation of the state vector for small angle ϵ is $R_z\psi(x, y, z) = \psi(x - \epsilon y, \epsilon x + y, z)$, which we can expand as the series

$$\psi(x + \Delta x, y + \Delta y, z) \approx \psi(x, y, z) + \frac{\partial \psi}{\partial x}\Delta x + \frac{\partial \psi}{\partial y}\Delta y + \text{higher order terms}$$

In this case, $\Delta x = -\epsilon y$ and $\Delta y = \epsilon x$, so we find

$$R_z\psi(x, y, z) \approx \psi(x, y, z) - \frac{\partial \psi}{\partial x} y + \frac{\partial \psi}{\partial y} x$$

which we can simplify to

$$R_z\psi(x, y, z) \approx \psi(x, y, z) + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(x, y, z)$$

and so

$$R_z\psi(x, y, z) \approx \left(I + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(x, y, z)$$

(c) This is just our coordinate representation for the angular momentum operator, so we can thus conclude that $\boxed{R_z\psi(x, y, z) \approx \left[1 + \frac{i}{\hbar}L_z\right]}$. In fact, for larger angles, it can be shown that the exact form of the rotation matrix is $R_z = e^{iL_z\hbar}$. The angular momentum operators are said to be the *generators* of the group of rotation matrices. They are actually elements of what's called a *Lie Algebra*, and the rotation matrices are the corresponding *Lie Group*.

* Note that I made the signs consistent here. In actual fact, the positive or negative sign simply defines the direction of the rotation, and isn't a big deal.

4. We know that $L_x = \frac{1}{2}(L_+ + L_-)$, and from class we determined that the corresponding matrix form is

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The corresponding eigenvalues and eigenvectors for L_x in terms of those for L_z can thus be pulled directly out of Maple:

$$\phi^x_{+1} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}\left(\phi_1 + \sqrt{2}\phi_2 + \phi_{-1}\right) \quad \text{Eigenvalue :}\hbar$$

$$\phi^x_{0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\phi_1 - \phi_{-1}\right) \quad \text{Eigenvalue :}0$$

$$\phi^x_{-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}\left(\phi_1 - \sqrt{2}\phi_2 + \phi_{-1}\right) \quad \text{Eigenvalue :}-\hbar$$