

Assignment #1, Physics 322

Due Date: Monday 08 Feb 2010

1. (a) Write down the wavefunction of a particle that is in an equal superposition of the (l, m) angular momentum states $(0, 0)$, $(1, 0)$, $(2, -1)$.

(b) Plot the corresponding probability density.

(c) Discuss the following limiting cases: (1) the probability of being in the $(0, 0)$ state is very small (but the other states are equally probable), (2) the probability of being in the $(0, 0)$ state is extremely high compared to the other two.

2. Verify the following commutation relations:

(a) $[L_i, R_j] = i\hbar\epsilon_{ijk}R_k$, where R_i are the position operators for the i^{th} coordinate (*i.e.* $R_1 = X$, *etc.*), and ϵ_{ijk} is the Levi-Civita tensor.

(b) $[L_i, P_j] = i\hbar\epsilon_{ijk}P_k$

(c) $[L_i, R^2] = [L_i, P^2] = [L_i, R \cdot P] = 0$

3. Recall that one can write $\cos a\theta + i \sin a\theta = e^{ia\theta}$, and hence the right-hand side of this equation may be thought of as a rotation. In fact, we define an *operator* which takes a vector and rotates it. This exercise will show you that quantum mechanical rotations are *generated* by the angular momentum operator, and that in general the rotation operator can be written as $R_z(\phi) = \exp(-i\phi L_z/\hbar)$.

(a) First, consider a classical rotation matrix, defined as $\mathcal{R}_z(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

that rotates vectors by an angle ϕ about the z-axis. Using matrix multiplication, show that in the limit of a small angle $\phi = \epsilon$, this rotation takes the vector (x, y, z) to $(x - \epsilon y, \epsilon x + y, z)$.

(b) Now, let's extend this to quantum state vectors. The above result means we can define a (quantum) rotation operator $R_z(\phi)$ that acts on the coordinate basis as $R_z(\phi)|xyz\rangle = |x'y'z'\rangle$. As in part (a), in the case of small angles, we get $R_z(\epsilon)|xyz\rangle = |x - \epsilon y, \epsilon x + y, z\rangle$.

The action of R_z on an arbitrary quantum state is just represented by $R_z|\psi\rangle$, so to understand what this does to the state, we must cast it into the position (coordinate) representation,

$$\langle xyz|R_z|\psi\rangle = \langle x - \epsilon y, \epsilon x + y, z|\psi\rangle$$

which is equivalent to

$$R_z\psi(x, y, z) = \psi(x - \epsilon y, \epsilon y + x, z)$$

Show that the coordinate representation of the angular momentum operator $(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$ can be derived from the right-hand side of the above expression [Hint: *Taylor* the solution to suit your needs...].

(c) Based on your answer to part (b), show that the rotation operator is $R_z(\epsilon) = I - \frac{i\epsilon}{\hbar} L_z$, and thus conclude that the angular momentum operator really is the generator of infinitesimal rotations.

4. The eigenfunctions for the operators L^2 and L_z with eigenvalues $l = 1$ and $m = +1, 0, -1$ are $\phi_{+1}, \phi_0, \phi_{-1}$. Determine the corresponding eigenfunctions and eigenvalues of L_x [Hint: use what you know about the action of L_+ and L_- on the eigenfunctions ϕ_k , and build the matrix form for L_x].