1. (a) Write down the wavefunction of a particle that is in an equal superposition of the \((l, m)\) angular momentum states \((0, 0), (1, 0), (2, -1)\).

(b) Plot the corresponding probability density.

(c) Discuss the following limiting cases: (1) the probability of being in the \((0, 0)\) state is very small (but the other states are equally probable), (2) the probability of being in the \((0, 0)\) state is extremely high compared to the other two.

2. Verify the following commutation relations:

(a) 
\[
[L_i, R_j] = i\hbar\epsilon_{ijk}R_k,
\]
where \(R_i\) are the position operators for the \(i\)th coordinate (i.e. \(R_1 = X\), etc...), and \(\epsilon_{ijk}\) is the Levi-Civita tensor.

(b) 
\[
[L_i, P_j] = i\hbar\epsilon_{ijk}P_k
\]

(c) 
\[
[L_i, R^2] = [L_i, P^2] = [L_i, R \cdot P] = 0
\]

3. Recall that one can write \(\cos a\theta + i\sin a\theta = e^{ia\theta}\), and hence the right-hand side of this equation may be thought of as a rotation. In fact, we define an operator which takes a vector and rotates it. This exercise will show you that quantum mechanical rotations are generated by the angular momentum operator, and that in general the rotation operator can be written as \(R_z(\phi) = \exp(-i\phi L_z/\hbar)\).

(a) First, consider a classical rotation matrix, defined as 
\[
R_z(\phi) = \begin{pmatrix} 
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1 
\end{pmatrix},
\]
that rotates vectors by an angle \(\phi\) about the z-axis. Using matrix multiplication, show that in the limit of a small angle \(\phi = \epsilon\), this rotation takes the vector \((x, y, z)\) to \((x - \epsilon y, \epsilon x + y, z)\).

(b) Now, let’s extend this to quantum state vectors. The above result means we can define a (quantum) rotation operator \(R_z(\phi)\) that acts on the coordinate basis as \(R_z(\phi)|xyz\rangle = |x'y'z'\rangle\). As in part (a), in the case of small angles, we get \(R_z(\epsilon)|xyz\rangle = |x - \epsilon y, \epsilon x + y, z\rangle\).

The action of \(R_z\) on an arbitrary quantum state is just represented by \(R_z|\psi\rangle\), so to understand what this does to the state, we must cast it into the position (coordinate) representation,

\[
\langle xyz|R_z|\psi\rangle = \langle x - \epsilon y, \epsilon x + y, z|\psi\rangle
\]

which is equivalent to

\[
R_z\psi(x, y, z) = \psi(x - \epsilon y, \epsilon x + y, z)
\]
Show that the coordinate representation of the angular momentum operator \((x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})\) can be derived from the right-hand side of the above expression [Hint: Taylor the solution to suit your needs...].

(c) Based on your answer to part (b), show that the rotation operator is \(R_z(\epsilon) = I - i\epsilon \frac{\hbar}{\hbar} L_z\), and thus conclude that the angular momentum operator really is the generator of infinitesimal rotations.

4. The eigenfunctions for the operators \(L^2\) and \(L_z\) with eigenvalues \(l = 1\) and \(m = +1, 0, -1\) are \(\phi_{+1}, \phi_0, \phi_{-1}\). Determine the corresponding eigenfunctions and eigenvalues of \(L_x\) [Hint: use what you know about the action of \(L_+\) and \(L_-\) on the eigenfunctions \(\phi_k\), and build the matrix form for \(L_x\)].