Assignment #1, Physics 322
Due Date: Wednesday 28 January 2009

1. Submit your results from the Jan 21st class worksheet.

2. Verify the following commutation relations:

   (a) $[L_i, R_j] = i\hbar\epsilon_{ijk}R_k$, where R_i are the position operators for the i^{th} coordinate (*i.e.* $R_1 = X$, *etc...*), and ϵ_{ijk} is the Levi-Civita tensor.

   (b) $[L_i, P_j] = i\hbar\epsilon_{ijk}P_k$

   (c) $[L_i, R^2] = [L_i, P^2] = [L_i, R \cdot P] = 0$

3. Recall that one can write $\cos a\theta + i \sin a\theta = e^{ia\theta}$, and hence the right-hand side of this equation may be thought of as a rotation. In fact, we define an *operator* which takes a vector and rotates it. This exercise will show you that quantum mechanical rotations are *generated* by the angular momentum operator, and that in general the rotation operator can be written as $R_z(\phi) = \exp(-i\phi L_z/\hbar)$.

   (a) First, consider a classical rotation matrix, defined as $\mathcal{R}_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, that rotates vectors by an angle ϕ about the z-axis. Using matrrix multiplication, show that in the limit of a small angle $\phi = \epsilon$, this rotation takes the vector (x, y, z) to $(x - \epsilon y, \epsilon x + y, z)$.

   (b) Now, let's extend this to quantum state vectors. The above result means we can define a (quantum) rotation operator $R_z(\phi)$ that acts on the coordinate basis as $R_z(\phi)|xyz\rangle = |x'y'z'\rangle$. As in part (a), in the case of small angles, we get $R_z(\epsilon)|xyz\rangle = |x - \epsilon y, \epsilon x + y, z\rangle$.

   The action of R_z on an arbitrary quantum state is just represented by $R_z|\psi\rangle$, so to understand what this does to the state, we must cast it into the position (coordinate) representation,

   $$\langle xyz|R_z|\psi\rangle = \langle x - \epsilon y, \epsilon x + y, z|\psi\rangle$$

   which is equivalent to

   $$R_z\psi(x, y, z) = \psi(x - \epsilon y, \epsilon y + x, z)$$

   Show that the coordinate representation of the angular momentum operator $(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ can be derived from the right-hand side of the above expression [Hint: *Taylor* the solution to suit your needs...].

   (c) Based on your answer to part (b), show that the rotation operator is $R_z(\epsilon) = I - \frac{i\epsilon}{\hbar}L_z$, and thus conclude that the angular momentum operator really is the generator of infinitessimal rotations.

4. The eigenfunctions for the operators L^2 and L_z with eigenvalues $l = 1$ and $m = +1, 0, -1$ are $\phi_{+1}, \phi_0, \phi_{-1}$. Determine the corresponding eigenfunctions and eigenvalues of L_x [Hint: use what you know about the action of L_+ and L_- on the eigenfunctions ϕ_k, and build the matrix form for L_x).