Assignment #7 solutions

1. See Maple sheet.

2. See Maple sheet.

3. The action of the ladder operators on the QHO states, in Dirac notation, is

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad , \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

The first thing to notice here is that the states $|n\rangle$ are *not* eigenstates of the operators $a, a^\dagger$. This implies that the matrix form of the ladder operators will be *off-diagonal*.

We can write the matrix elements in Dirac notation as follows:

$$a_{ij} = \langle j|a|i\rangle \quad , \quad a^\dagger_{ij} = \langle j|a^\dagger|i\rangle$$

From the action defined above, we know that (for a):

$$\langle j|a|i\rangle = \sqrt{i}\langle j|i-1\rangle$$

This can only be non-zero when $j = i-1$. Hence, the non-zero matrix elements will be $a_{21} = \sqrt{2}, a_{32} = \sqrt{3},$, *etc...*:

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \\ & & \vdots & & \ddots \end{pmatrix}$$

We can test this by evaluating $a|0\rangle$, which we know should be zero:

$$a|0\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \\ & & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = 0$$

Perfect! Now for $a^\dagger$: by definition, this is the Hermitian transpose of this matrix:

$$a^\dagger = (a^*)^T = \begin{pmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \\ 0 & 0 & \sqrt{3} & \\ & \vdots & & \ddots \end{pmatrix}$$

Again as a double-check, we should find, for example, $a^\dagger|1\rangle = \sqrt{2}|2\rangle$:

$$a^\dagger|1\rangle = \begin{pmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \\ 0 & 0 & \sqrt{3} & \\ & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ \vdots \end{pmatrix}$$

which is true!

(b) The Hamiltonian is $H = N + \frac{1}{2}$, which must be Hermitian and diagonal. The matrix representations of $a, a^\dagger$ themselves are off-diagonal, but it's easy to see that their product is in fact diagonal:

$$a^\dagger a = \begin{pmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \\ 0 & 0 & \sqrt{3} & \\ & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \\ & & \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \\ 0 & 0 & 0 & 3 & \\ & \vdots & & & \ddots \end{pmatrix}$$

It's easy to see that this matrix gives $a^\dagger a|n\rangle = n|n\rangle$, and thus the Hamiltonian is:

$$H = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \\ 0 & \frac{3}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{5}{2} & 0 & \\ 0 & 0 & 0 & \frac{7}{2} & \\ & \vdots & & & \ddots \end{pmatrix}$$

(c) The matrix above is obviously Hermitian, since $N^\dagger = N$. We know this because the diagonal elements are real, and off-diagonal are 0.