Assignment #6 solutions

1. Almost everyone got this, so no solutions needed.

2. It's straightforward to calculate $\rho = \sqrt{2m|E - V_0|\hbar^2} = \sqrt{2(9 \times 10^{-31})(1.6 \times 10^{-19})}(10^{-34})^2 \approx 5 \times 10^9$ m^{-1}, so $1/\rho \approx 2 \times 10^{-10}$ m. Since $a \sim 10^{-10}$ m, we have $\rho > a$, and thus the particle will likely tunnel.

   (b) The tunneling probability is easily calculated from the formula provided, and we find $T \approx 0.65$, or a 65% chance of tunneling.

3. See Maple worksheet.

4. (a) Using the fact that $a = \frac{1}{\sqrt{2}}[\hat{X} + i\hat{P}$, and $[\hat{X}, \hat{P}] = i$, we can work out the first commutator:

$$[a, a^\dagger] = \frac{1}{2}[\hat{X} + i\hat{P}, \hat{X} - i\hat{P}]$$
$$= \frac{1}{2}(-i[\hat{x}, \hat{P}] - i[\hat{X}, \hat{P}])$$
$$= \frac{1}{2}(-i)(2i)$$
$$[a, a^\dagger] = 1$$

   (b) Since $N = a^\dagger a$, we have $[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger a a^\dagger - a^\dagger a^\dagger a$. We can use the commutativity relation to interchange two of the operators in the first term:

$$aa^\dagger - a^\dagger a = 1 \implies aa^\dagger = a^\dagger a + 1$$

   and so $\boxed{[N, a^\dagger] = a^\dagger(a^\dagger a + 1) - a^\dagger a^\dagger a = a^\dagger}$

   (c) Note that $[N, a] = [a^\dagger a, a] = a^\dagger a a - a a^\dagger a = (a^\dagger a - a a^\dagger)a$. Since we know from part (a) that $[a, a^\dagger] a a^\dagger - a^\dagger a = 1$, the commutator in question can immediately be solved as $\boxed{[N, a] = -a}$