Assignment #6, Physics 321
Due Date: Monday 16 November 2009

For most of the questions on this assignment, it will greatly ease your burden to use Maple. If you still don't know how to use it, ask me to show you ASAP.

1. Using the Maple application we saw in class (available for download from the website) for a time-dependent solution to Schrödinger's Equation, argue the following (provide print-outs of your plots to justify your answers):

   (a) When a wavepacket of energy E encounters a potential step of energy $V_0 > E$, the wavepacket will undergo reflection.

   (b) When a wavepacket of energy E encounters an *attractive* potential well of energy $-V_0 < E$ whose width a is much less than the width of the wavepacket, the particle will generally scatter (and not be captured by the potential). Assume the wavepacket has width 2, which is roughly the "full-width-half-maximum" (FWHM) of the distribution. This simulates, for example, what might happen at the LHC if a mini-black hole is produced that is smaller than most particles (*i.e.* low probability of absorption).

   (c) When a wavepacket of energy E encounters a *very* attractive potential well of energy $-V_0 \ll E$ whose width a is greater than the width of the wavepacket, the particle will either scatter or become trapped like a "particle in a box".

2. In quantum tunneling, a free particle of energy E encountering a potential step $V(x) = V_0, x \in [0, a]$, where $E < V_0$, has a finite probability of "breaking" through the barrier. We saw in class that the tunneling probability is

$$T = \frac{1}{1 + \left(\frac{k^2+\rho^2}{2k\rho}\right)^2 \sinh^2(\rho a)}$$

   where k is the momentum outside the barrier. Because of the $\sinh(\rho a)$ (*i.e.* exponential) dependence of the probability, we can extract a physically meaningful interpretation of the likelihood of tunneling for a barrier of a particular thickness. Inside the barrier, the wavefunction is known as an *evanescent wave* – a wave that exponentially decreases with distance – and its extent can be defined by a *range*, defined to be $r \sim 1/\rho$. If $a \leq r$, the particle stands a good chance of tunneling, while if $a \gg r$, the particle will have a small tunneling probability.

(a) Suppose an electron of energy 1 eV (1.6×10^{-19} J) is confined by an electric field, such that the potential energy between the two is twice as large as the energy of the electron, $V_0 = 2E_0$. The field has a thickness of $1\mathring{A} = 10^{-10}$ m. Determine the range of the wavefunction in the potential barrier, and decide whether or not the probability of tunneling is high or low.

(b) Calculate the tunneling probability of the electron. Recall that $k = \sqrt{\frac{2mE}{\hbar^2}}$, where E is the particle's energy and m the mass, and $\rho = \sqrt{\frac{2m|E-V_0|}{\hbar^2}}$.

3. Solving the Schrödinger equation gives the QHO eigenfunctions

$$\phi_n(x) = \frac{1}{\sqrt{n!2^n}}\sqrt{\frac{\beta}{\sqrt{\pi}}}e^{-\beta^2x^2/2}H_n(\beta x)$$

where $\beta = \sqrt{\frac{m\omega}{\hbar}}$, and $H_n(y)$ are the Hermite polynomials. For $n = 0, 1, 2, 3$, show that wavefunction $\phi_n(x)$ is...

(a) ... normalized.

(b) ... orthogonal to $\phi_m(x)$ for $m \neq n$.

The Hermite polynomials are defined in Maple by typing $with(orthopoly)$, and are called as $H(n, y)$, where y is the input given in the equation, $y = \beta x$.

4. The operator $a = \frac{1}{\sqrt{2}}[\hat{X} + i\hat{P}]$ is called the "lowering" operator (or the annihilation operator). Derive the following commutation relations for a and $a^\dagger$:

(a) $[a, a^\dagger]$

(b) $[a^\dagger a, a]$

(c) $[a^\dagger a, a^\dagger]$