Assignment #5 solutions, Physics 321

1. First, note that we can simplify the wavefunction as

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(e^{iE_1t/\hbar}\phi_1(x) + e^{iE_2t/\hbar}\phi_2(x)\right) = \left(\sin(\pi x) + e^{(E_2-E_1)\hbar/t}\sin(2\pi x)\right)e^{iE_1t/\hbar}$$

by factoring out a global phase. This will make the algebra simpler when we calculate the current, since we'll be multiplying complex conjugates (and the phase will disappear).

Setting $\Delta E = E_2 - E_1$, the components of the probability current are:

$$\psi^*\nabla_x\psi = (\sin(\pi x) + e^{-i\Delta Et/\hbar}\sin(2\pi x))(\pi\cos(\pi x) + 2\pi e^{+i\Delta Et/\hbar}\cos(2\pi x))$$

$$\psi\nabla_x\psi^* = (\sin(\pi x) + e^{i\Delta Et/\hbar}\sin(2\pi x))(\pi\cos(\pi x) + 2\pi e^{-i\Delta Et/\hbar}\cos(2\pi x))$$

Since $\mathcal{J} \sim \psi^*\nabla_x\psi - \psi\nabla_x\psi^*$, all purely real terms in the expansion of these components will cancel in the probability current, and the only ones left over will be those with $e^{\pm i\Delta Et/\hbar}$. These are the cross-terms, which differ only in the sign of the complex exponential:

$$\psi^*\nabla_x\psi \longrightarrow \pi\cos(\pi x)\sin(2\pi x)e^{+i\Delta Et/\hbar} + 2\pi\sin(\pi x)\cos(2\pi x)e^{-i\Delta Et/\hbar}$$

$$\psi\nabla_x\psi^* \longrightarrow \pi\cos(\pi x)\sin(2\pi x)e^{-i\Delta Et/\hbar} + 2\pi\sin(\pi x)\cos(2\pi x)e^{+i\Delta Et/\hbar}$$

So, when we evaluate the probability current, we get

$$\begin{aligned}\mathcal{J}(x,t) = \frac{\hbar\pi}{2mi}&\left\{\cos(\pi x)\sin(2\pi x)\left(e^{+i\Delta Et/\hbar} - e^{-i\Delta Et/\hbar}\right)\right\}\\ &-2\left\{\cos(2\pi x)\sin(\pi x)\left(e^{+i\Delta Et/\hbar} - e^{-i\Delta Et/\hbar}\right)\right\}\end{aligned}$$

Since $\sin y = \frac{e^y - e^{-y}}{2i}$, this simplifies to

$$\boxed{\mathcal{J}(x,t) = \frac{\hbar\pi}{m}\left(\cos(\pi x)\sin(2\pi x) - 2\sin(\pi x)\cos(2\pi x)\right)\sin\left(\frac{\Delta Et}{\hbar}\right)}$$

So, it *is* periodic in time, as we saw.

(b) In order to verify the continuity equation, $\frac{\partial\rho}{\partial t} + \nabla\cdot\mathcal{J} = 0$, all we need to do is find the time derivative of the probability density, and the spatial derivative of the probability current. In the first case, we know

$$\rho(x,t) = \frac{1}{2}\phi_1(x)^2 + \frac{1}{2}\phi_2(x)^2 + \phi_1(x)\phi_2(x)\cos[(\omega_2 - \omega_1)t]$$

where $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, so we get

$$\frac{\partial\rho}{\partial t} = -\frac{2\Delta E}{\hbar}\sin(\pi x)\sin(2\pi x)\sin(\Delta Et/\hbar)$$

Since $E = \frac{\hbar^2\pi^2 n^2}{2m}$, we have $\Delta E = \frac{4\hbar^2\pi^2}{2m} - \frac{\hbar^2\pi^2}{2m} = \frac{3\hbar^2\pi^2}{2m}$, so

$$\frac{\partial\rho}{\partial t} = -\frac{3\hbar\pi^2}{m}\sin(\pi x)\sin(2\pi x)\sin\left(\frac{\Delta E t}{\hbar}\right)$$

The divergence of $\mathcal{J}$ is easy to calculate, since we're only dealing with one dimension:

$$\nabla_x\left(\cos(\pi x)\sin(2\pi x)\right) = \pi(-\sin(\pi x)\sin(2\pi x) + 2\cos(\pi x)\cos(2\pi x))$$

$$\nabla_x\left(\sin(\pi x)\cos(2\pi x)\right) = \pi(\cos(\pi x)\cos(2\pi x) - 2\sin(\pi x)\sin(2\pi x))$$

and thus

$$\begin{aligned}\nabla_x\mathcal{J} &= \frac{\hbar\pi}{m}\nabla_x\left(\cos(\pi x)\sin(2\pi x) - 2\sin(\pi x)\cos(2\pi x)\right)\sin\left(\frac{\Delta E t}{\hbar}\right)\\ &= \frac{\hbar\pi^2}{m}(-\sin(\pi x)\sin(2\pi x) + 2\cos(\pi x)\cos(2\pi x)\\ &\quad -2\cos(\pi x)\cos(2\pi x) + 4\sin(\pi x)\sin(2\pi x))\sin\left(\frac{\Delta E t}{\hbar}\right)\\ \nabla_x\mathcal{J} &= \frac{3\hbar\pi^2}{m}\sin(\pi x)\sin(2\pi x)\sin\left(\frac{\Delta E t}{\hbar}\right)\end{aligned}$$

Phew! Sure enough, we have shown that $\frac{\partial\rho}{\partial t} + \nabla\cdot\mathcal{J} = 0$! Horray!

2. (a) In this case, the wavefunctions in the two regions are

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}, \quad x < 0$$

$$\psi_{II}(x) = Fe^{-\rho x}, \quad x \geq 0$$

where $\psi_{II}(x)$ is now exponentially decaying, since the momentum is now complex.

As always, we require the function to be both continuous and differentiable at the potential boundary, so:

$$\begin{aligned}\psi_I(0) &= \psi_{II}(0)\\ A + B &= F\end{aligned}$$

and

$$\begin{aligned}\psi_I(0) &= \psi_{II}(0)\\ ik(A - B) &= -\rho F\\ \frac{-ik}{\rho}(A - B) &= F\end{aligned}$$

Solving for A in terms of B, we get

$$\begin{aligned} A+B &= \frac{-ik}{\rho}(A-B) \\ \frac{B}{A} &= \frac{ik+\rho}{ik-\rho} \end{aligned}$$

Note that we actually could have derived this expression from the case where $V_0 < E$, simply by making the substitution $k_2 = i\rho$ (it's the same math at work!).

Since $|B/A|^2 = (B/A)(B/A)^*$, the reflection probability is
$\boxed{R = \left(\frac{\rho-ik}{\rho+ik}\right)\left(\frac{\rho+ik}{\rho-ik}\right) = 1}$
as expcted. The transmission probability is thus $T = 1 - R = 0$. The particle never travels through the potential step, but always reflects.

3. (a) If the classical Hamiltonian is $\mathcal{H} = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$, then using the replacement $p \to P, x \to X$ we obtain the Hamiltonian operator $\boxed{H = \frac{1}{2m}P^2 - \frac{1}{2}kX^2}$.

   (b) By Ehrenfest's theorem, the time rate-of-change of the position expectation value is

   $$\frac{d}{dt}\langle X\rangle = \frac{1}{i\hbar}\langle [X,H]\rangle$$

   where the commutator is

   $$[X,H] = \frac{1}{2m}[X,P^2] - \frac{1}{2}k[X,X^2] = \frac{1}{2m}\left(P[X,P] + [X,P]P\right) + 0 = \frac{i\hbar}{m}P$$

   Thus, $\boxed{\frac{d}{dt}\langle X\rangle = \frac{1}{i\hbar}\frac{i\hbar}{m}\langle P\rangle = \frac{\langle P\rangle}{m}}$, which is classical velocity.

   Similarly, we can see that

   $$\frac{d}{dt}\langle P\rangle = \frac{1}{i\hbar}\langle [P,H]\rangle$$

   with

   $$[P,H] = \frac{1}{2m}[P,P^2] - \frac{1}{2}k[P,X^2] = -\frac{1}{2}k\left(X[P,X] + [P,X]X\right) = -i\hbar kX$$

   So, $\boxed{\frac{d}{dt}\langle P\rangle = \frac{i\hbar k}{i\hbar}\langle X\rangle = -k\langle X\rangle}$, which we recognize as Hooke's Law for harmonic oscillators (springs!).

4. (a) The neutrino mass eigenstates form the basis $\{|m_1\rangle, |m_2\rangle\}$, and thus we can make the association $|m_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|m_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A general mass eigenstate can be written as $|\nu_m\rangle = |\nu_1\rangle + |\nu_2\rangle$ (we'll ignore normalization for the moment).

The rotation matrix thus mixes the states according to the rules of matrix multiplication,

$$|\nu_e\rangle = R(\theta)|\nu_1\rangle = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$|\nu_\mu\rangle = R(\theta)|\nu_2\rangle = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

This means that $\boxed{|\nu_e\rangle = \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle}$ and $\boxed{|\nu_\mu\rangle = \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle}$.

(b) The time-evolution of the mass states is determined by the energy E_i, so we can write

$$|\nu_1(L)\rangle = e^{\frac{-im_1^2cL}{2E\hbar}}|\nu_1\rangle \quad ; \quad |\nu_2(L)\rangle = e^{\frac{-im_2^2cL}{2E\hbar}}|\nu_2\rangle$$

So, at distance L, the neutrino flavor state will be a mixture of these $L-$dependent states. For the $|\nu_\mu\rangle$ state in particular,

$$|\nu_\mu(L)\rangle = \sin\theta|\nu_1(L)\rangle + \cos\theta|\nu_2(L)\rangle = \sin\theta e^{\frac{-im_1^2cL}{2E\hbar}}|\nu_1\rangle + \cos\theta\frac{-im_2^2cL}{2E\hbar}|\nu_2\rangle$$

The probability that a ν_e turns into a ν_μ after travelling a distance L is thus $P_{e\to\mu} = |\langle\nu_\mu(L)|\nu_e(0)\rangle|^2$, where

$$\begin{aligned} \langle\nu_\mu(L)|\nu_e(0)\rangle &= \left[\langle\nu_1|\sin\theta e^{\frac{+im_1^2cL}{2E\hbar}} + \langle\nu_2|\cos\theta e^{\frac{+im_2^2cL}{2E\hbar}}\right][\cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle] \\ &= \sin\theta\cos\theta e^{\frac{+im_1^2cL}{2E\hbar}} - \sin\theta\cos\theta e^{\frac{+im_2^2cL}{2E\hbar}} \\ &= \frac{1}{2}\sin(2\theta)\left(e^{\frac{+im_1^2cL}{2E\hbar}} - e^{\frac{+im_2^2cL}{2E\hbar}}\right) \end{aligned}$$

The complex conjugate of this expression is

$$\langle\nu_e(0)|\nu_\mu(L)\rangle = \frac{1}{2}\sin(2\theta)\left(e^{\frac{-im_1^2cL}{2E\hbar}} - e^{\frac{-im_2^2cL}{2E\hbar}}\right)$$

so the probability is

$$|\langle\nu_\mu(L)|\nu_e(0)\rangle = \frac{1}{4}\sin^2(2\theta)\left(2 - e^{\frac{+i(m_1^2-m_2^2)cL}{2E\hbar}} - e^{\frac{-i(m_1^2-m_2^2)cL}{2E\hbar}}\right)$$

Using the fact that $\cos\alpha = \frac{e^{ia}+e^{-i\alpha}}{2}$ and the second identity given in the question, this can be re-written as $\boxed{P_{\nu_e\to\nu_\mu} = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2cL}{4E\hbar}\right)}$, which is what we wanted!