

## Assignment #4

### Physics 321

1. (a) We can easily show that

$$\psi(x) = c_1 e^{\frac{ipx}{\hbar}} + c_2 e^{-\frac{ipx}{\hbar}}$$

satisfies the differential part of the free-particle Schrödinger equation by “plug-and-chug”:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} (c_1 e^{ipx/\hbar}) + -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} (c_2 e^{-ipx/\hbar})$$

where each derivative gives:

$$\begin{aligned} \frac{d^2}{dx^2} (c_1 e^{ipx/\hbar}) &= -\frac{p^2}{\hbar^2} (c_1 e^{ipx/\hbar}) \\ \frac{d^2}{dx^2} (c_2 e^{-ipx/\hbar}) &= -\frac{p^2}{\hbar^2} (c_2 e^{-ipx/\hbar}) \end{aligned}$$

So, we have

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = \frac{\hbar^2}{2m} \frac{p^2}{\hbar^2} \psi(x) = \frac{p^2}{2m} \psi(x)$$

(b) From the definitions  $\sin(\alpha x) = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}$  and  $\cos(\alpha x) = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}$ , we can determine that

$$\begin{aligned} \psi(x) &= A \left( \frac{e^{ipx/\hbar} - e^{-ipx/\hbar}}{2i} \right) + B \left( \frac{e^{ipx/\hbar} + e^{-ipx/\hbar}}{2} \right) \\ \psi(x) &= \left( \frac{A}{2i} + \frac{B}{2} \right) e^{ipx/\hbar} + \left( \frac{B}{2} - \frac{A}{2i} \right) e^{-ipx/\hbar} \\ \psi(x) &= \frac{1}{2} (B - Ai) e^{ipx/\hbar} + \frac{1}{2} (B + Ai) e^{-ipx/\hbar} \end{aligned}$$

So, we see that  $\boxed{c_1 = \frac{1}{2} (B - Ai), c_2 = \frac{1}{2} (B + Ai)}$ . In fact, notice that  $c_1 = c_2^*$  – they are complex conjugates of one another! So, that gives a new meaning to the “act” of complex conjugation! Recall that the two solutions are waves traveling to the left and right, respectively. Since both functions are complex conjugates of each other, as are the coefficients, then complex conjugation must have something to do with a physical reversal of *direction*! Cool! An otherwise abstract concept has real meaning! (no pun intended).

(c) From part (a) and the Schrödinger equation, we know that

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = E \psi(x)$$

and thus we can make the association  $\boxed{E = \frac{p^2}{2m}}$ , which is just the kinetic energy for a particle of momentum  $p$ .

2. First, we determine the normalization constant. We know that

$$1 = \langle \psi | \psi \rangle = \int_0^1 |\psi(x)|^2 dx = \int_0^1 A^2 x^4 dx$$

and so evaluating this integral gives

$$1 = \frac{A^2}{5} \implies A = \sqrt{5}$$

In order to evaluate the HUP, we'll need to calculate the expectation values of the position and momentum operators,  $\langle X \rangle$  and  $\langle P \rangle$ , as well as the expectations of their squares,  $\langle X^2 \rangle$  and  $\langle P^2 \rangle$ . This is because the uncertainties in position and momentum are

$$\Delta x = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}, \quad \Delta p = \sqrt{\langle P^2 \rangle - \langle P \rangle^2}$$

Recalling that

$$\langle X \rangle = \langle \psi | X | \psi \rangle = \int \langle \psi | x \rangle \langle x | X | \psi \rangle dx = \int_0^1 x |\psi(x)|^2 dx$$

we can evaluate this integral easily to give

$$\langle X \rangle = 5 \int_0^1 x^5 dx = \frac{5}{6}$$

Similarly,

$$\langle X^2 \rangle = \int_0^1 x^2 |\psi(x)|^2 dx = 5 \int_0^1 x^6 dx = \frac{5}{7}$$

Thus, the uncertainty in position is

$$\Delta x = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\frac{5}{7} - \frac{25}{36}} = 0.14$$

So far so good – but what about momentum? Looking at  $\langle P \rangle$  and  $\langle P^2 \rangle$ , we find:

$$\langle P \rangle = \int \psi^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi(x) dx$$

which gives

$$\langle P \rangle = \frac{10\hbar}{i} \int_0^1 x^3 dx = \frac{5\hbar}{2i}$$

The other term is

$$\langle P^2 \rangle = -\hbar^2 \int \psi^*(x) \frac{d^2}{dx^2} \psi(x) dx$$

$$\langle P^2 \rangle = -10\hbar^2 \int_0^1 x^2 dx = -\frac{10\hbar^2}{3}$$

and thus

$$\Delta p = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \sqrt{-\frac{10\hbar^2}{3} + \frac{25\hbar^2}{4}} = \hbar\sqrt{\frac{35}{12}} = 1.71\hbar$$

So, we get  $\boxed{(\Delta x)(\Delta p) = 0.24\hbar}$ , which violates the HUP. So,  $\psi(x)$  cannot represent a real wavefunction, at least in the range  $0 \leq x \leq 1$ .

3. See Maple solutions!
4. See Maple solutions again!