

Assignment #3 solutions

Physics 321

1. (a) Let's act the commutator on the state $|\psi\rangle$, and we'll assume that $|\psi\rangle$ is a simultaneous eigenstate of A and B . That is, $A|\psi\rangle = a|\psi\rangle$, and $B|\psi\rangle = b|\psi\rangle$. Since we are told these are Hermitian, we know that $a, b \in \mathbb{R}$. So,

$$[A, B]|\psi\rangle = (AB - BA)|\psi\rangle = AB|\psi\rangle - BA|\psi\rangle$$

Paying attention to the order of operations, we have

$$AB|\psi\rangle = b(A|\psi\rangle) = ab|\psi\rangle \quad , \quad BA|\psi\rangle = a(B|\psi\rangle) = ab|\psi\rangle$$

So, $[A, B]|\psi\rangle = ab - ab = 0$! The commutator of two simultaneous observables is real, but is also vanishes. We'll see why this is important, and what its physical implications are, when we get into two- and three-dimensional solutions to the Schrödinger equation!

(b) Let U be a unitary operator, and $|\phi\rangle = U|\psi\rangle$. Taking the hermitian conjugate of this expression, we find $\langle\psi|U^\dagger = \langle\phi|$. Since we are told that U preserves the norm of the vectors, we know

$$\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = \langle\psi|U^\dagger U|\psi\rangle$$

In order for this to be true, U must satisfy the condition $U^\dagger U = U U^\dagger = \mathbb{I}$. This is the definition of a unitary matrix. It can alternatively be written $U^\dagger = U^{-1}$.

2. If $\langle\gamma|\gamma\rangle \geq 0$, and $|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle$, then we can prove the Schwarz Inequality as follows:

$$\begin{aligned} \langle\gamma|\gamma\rangle &= \left[\langle\beta| - \frac{\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle}\langle\alpha| \right] \left[|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle \right] \\ &= \langle\beta|\beta\rangle + \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{|\langle\alpha|\alpha\rangle|^2}\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} - \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle} \\ &= \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle} \end{aligned}$$

since $\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = |\langle\alpha|\beta\rangle|^2$. This quantity must be ≥ 0 , we have

$$\begin{aligned} \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle} &\geq 0 \\ \langle\beta|\beta\rangle &\geq \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle} \\ \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle &\geq |\langle\alpha|\beta\rangle|^2 \end{aligned}$$

which is the Schwarz Inequality!!

3. (a) It's straightforward to show that $R(\theta)R(\phi) = R(\phi)R(\theta)$:

$$R(\theta)R(\phi) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} =$$

$$\begin{pmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & \cos \theta \sin \phi + \sin \theta \cos \phi \\ -\sin \theta \cos \phi - \cos \theta \sin \phi & \sin \theta \sin \phi + \cos \theta \cos \phi \end{pmatrix}$$

and

$$R(\phi)R(\theta) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} =$$

$$\begin{pmatrix} \cos \phi \cos \theta - \sin \phi \sin \theta & \cos \phi \sin \theta + \sin \phi \cos \theta \\ -\sin \phi \cos \theta - \cos \phi \sin \theta & \sin \phi \sin \theta + \cos \phi \cos \theta \end{pmatrix}$$

So, $\boxed{R(\theta)R(\phi) - R(\phi)R(\theta) = 0}$, and thus the rotations commute. The physical explanation is that, if you rotate something in the same plane by two consecutive angles, the total rotation will be the same regardless of the order. For example, the hands of a clock travel 90° whether you wait 10 minutes (30°), then 20 minutes (60°), or 20 minutes then 10 minutes.

(b) Everyone got this, so solutions won't be written up. The basic premise is that consecutive rotations in *different* planes will not result in the same angular displacement!

4. (a) The probability distribution $p(\theta)$ for the needle must satisfy

$$\int_0^{180} p(\theta) d\theta = 1$$

Since every point on the dial is *equally likely*, the probability distribution must be constant, so $p(\theta) = \frac{1}{180}$. Thus, the probability that it will land around 90° is the same as if it were to land around 45° , since the distribution is completely symmetric.

(b) The mean value of theta is simply

$$\bar{\theta} = \int_0^{180} \theta p(\theta) d\theta = \int_0^{180} \frac{\theta}{180} d\theta = \frac{1}{2(180)}[180^2 - 0] = 90^\circ$$

Of course, it should land, on average, right in the middle – the distribution is symmetric about this point!