Assignment #3

Physics 321

Due: Wednesday 30 September 2009

Answer all questions fully, describing each step you take. All questions are of equal value.

1. (a) The commutator of two operators is defined as $[A, B] = AB - BA$, which is also an operator. In general $[A, B] \neq 0$ (although there are special cases where this is true). Show that if the operators A and B are hermitian, then so is their commutator . (Hint: Remember that to show this, we have to act it on a state vector – $[A, B]|\psi\rangle$).

   (b) A *unitary* operator is one that preserves probabilities (and hence probability amplitudes) in a superposition. Suppose that U is a unitary operator that transforms the state $|\psi\rangle$ into the state $|\phi\rangle = U|\psi\rangle$. We thus require that $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle$. Derive a relationship between U and $U^\dagger$ that makes this true (and hence by doing so, derive the defining characteristic of a unitary matrix).

2. Derive the *Schwarz Inequality* $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$ for arbitrary vectors $|\alpha\rangle$ and $|\beta\rangle$. Since $|\alpha\rangle$ and $|\beta\rangle$ are arbitrary states, you cannot assume that they are normalized, *i.e.* $\langle\alpha|\alpha\rangle \neq 1$ and $\langle\beta|\beta\rangle \neq 1$. Fiurthermore, they are not necessarily orthogonal, $\langle\alpha|\beta\rangle \neq 0$. [Hint: define a new state $|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle$ and assume $\langle\gamma|\gamma\rangle \geq 0$]

3. Consider the rotation operator in two dimensions

$$R(\theta) = \begin{pmatrix} cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

   (a) Show that two successive rotations $R(\theta)$,$R(\phi)$ commute.

   (b) There are three rotation matrices in three dimensions, depending on the axis of rotation. They are:

$$R_x(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix}$$

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that any two successive rotations about different axes do not commute.

You can practice both types of rotations – two and three-dimensional – yourself, by rotating an object accordingly! The non-commutability of rotations isn't always quantum mechanics phenomena, but rather a property of the *group* that govern rotations. We'll discuss group theory a bit later in the course.

4. (a) Suppose a speedometer needle is free to swing between the angles $\theta = 0$ to $\theta = 180°$ (obviously not a very efficient speedometer). Determine the probability that, at some point, the speedometer will land at $\theta = 90°$. How would your answer be different for $\theta = 45°$?

   (b) Determine the mean value of θ (the expectation value). Does this answer make sense to you? Explain.