Assignment #2 solutions

Physics 321

1. (a) The wavefunction is problemtic, because the sum of the probability amplitudes squared does not equal unity: $(0.2)^2 + (0.4)^2 + (0.8)^2 = 0.84 \neq 1$.

   (b) The correct amplitude for ψ_1 must be $c_1 = \sqrt{1 - (0.4)^2 - (0.8)^2} = 0.45$, so the true state is

   $$\psi = 0.45\psi_1 + 0.4\psi_2 + 0.8\psi_3$$

   (c) If each state ψ_j is multiplied by a complex phase $e^{i\alpha_j}$, then the probabilities *do not change*. For example, each amplitude would be *e.g.*

   $$c_1 = \psi_1 \cdot \psi = \psi_1 \cdot \left[0.45e^{i\alpha_1}\psi_1 e^{i\alpha_1} + 0.4e^{i\alpha_2}\psi_2 e^{i\alpha_2} + 0.8e^{i\alpha_3}\psi_3 e^{i\alpha_3}\right] = 0.45\, e^{i\alpha_1}$$

   So, this means the probability is $|c_1|^2 = |c_1^* c_1| = (0.45e^{-i\alpha})(0.45e^{i\alpha}) = (0.45)^2 = 0.2$ The same is true for the rest.

2. (a) $\langle\alpha|\beta\rangle$: Complex number.
   (b) $|\alpha\rangle\langle\alpha|\beta\rangle$: A vector, since the term $|\alpha\rangle\langle\alpha|$ is a matrix, and $|\beta\rangle$ is a vector. Alternatively, we could say $\langle\alpha|\beta\rangle$ is a number multiplying the vector $|\alpha\rangle$.
   (c) $|\alpha\rangle\langle\beta|$: A matrix, since it is the outer product of two vectors.
   (d) $|\langle\alpha|\beta\rangle|^2$: A real number (specifically the transition probability between states $|\beta\rangle$ and $|\alpha\rangle$.)
   (e) $|\alpha\rangle\langle\alpha|$: A matrix (specifically the identity, if $|\alpha\rangle$ is normalized).

3. (a) If $z = 1 + e^{i\theta}$, then we can calculate z^* by substituting $i \longrightarrow -i$. Thus,

   $$z^* = 1 + e^{-i\theta}$$

   The quantity $z^2 = zz$, so

   $$z^2 = (1 + e^{i\theta})(1 + e^{i\theta}) = 1 + 2e^{i\theta} + e^{2i\theta}$$

   Note that this is different from $|z|^2 = z^*z = zz^*$, which is a real number:

   $$|z|^2 = (1 + e^{i\theta})(1 + e^{-i\theta}) = 1 + e^{i\theta} + e^{-i\theta} + e^{i\theta - i\theta}$$

   (b) Again, this is real, since the term $e^{i\theta - i\theta} = 1$, and $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ (recall the definition of $\cos\theta$:

   $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

   So, we get

   $$|z|^2 = 2 + 2\cos\theta$$

4. (a) Substituting in the definitions of the states $|L\rangle, |R\rangle$ into the expression for the polarization state $|\theta\rangle$ gives

$$\begin{aligned}
|\theta\rangle &= \frac{1}{\sqrt{2}}\left[e^{i\theta}|L\rangle + e^{-i\theta}|R\rangle\right] \\
&= \frac{1}{2}\left[e^{i\theta}[|x\rangle + i|y\rangle] + e^{\imath\theta}[|x\rangle - i|y\rangle]\right] \\
&= \frac{1}{2}\left[(e^{i\theta} + e^{-i\theta})|x\rangle + i(e^{i\theta} - e^{-i\theta})|y\rangle\right] \\
|\theta\rangle &= \cos\theta|x\rangle - \sin\theta|y\rangle
\end{aligned}$$

by using the complex definitions of $\sin\theta$ and $\cos\theta$.

(b) Since we require $\langle x|x\rangle = \langle y|y\rangle = 1$ and $\langle x|y\rangle = 0$, it is straightforward to evaluate the inner products $\langle L|L\rangle, \langle R|R\rangle$, and $\langle L|R\rangle$:

$$\begin{aligned}
\langle L|L\rangle &= \frac{1}{2}\left[(\langle x| - i\langle y|)(|x\rangle + i|y\rangle)\right] \\
&= \frac{1}{2}\left[\langle x|x\rangle + \langle y|y\rangle + i\langle x|y\rangle - i\langle y|x\rangle\right] \\
\langle L|L\rangle &= \frac{1}{2}\left[1 + 1 + 0 + 0\right] = 1
\end{aligned}$$

$$\begin{aligned}
\langle R|R\rangle &= \frac{1}{2}\left[(\langle x| + i\langle y|)(|x\rangle - i|y\rangle)\right] \\
\langle R|R\rangle &= \frac{1}{2}\left[1 + 1 + 0 + 0\right] = 1
\end{aligned}$$

which is the same calculation as $\langle L|L\rangle$. Finally,

$$\begin{aligned}
\langle L|R\rangle &= \frac{1}{2}\left[(\langle x| - i\langle y|)(|x\rangle - i|y\rangle)\right] \\
&= \frac{1}{2}\left[\langle x|x\rangle - \langle y|y\rangle - i\langle x|y\rangle - i\langle y|x\rangle\right] \\
\langle L|R\rangle &= \frac{1}{2}\left[1 - 1 + 0 + 0\right] = 0
\end{aligned}$$

So, the expected orthogonality constraints hold for these states too.