Assignment #1 solutions
Physics 321

1.

$$c = 2.99 \times 10^{8}\ \frac{\text{m}}{\text{s}} \quad ; \quad \hbar = 1.05 \times 10^{-34}\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}} \quad ; \quad G = 6.67 \times 10^{-11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$$

Using these fundamental constants, we can form quantities having specific units by raising each to a particular power: $\hbar^\alpha c^\beta G^\omega$, which will give a unit equation of the form

$$\left(\frac{\text{kg}\cdot\text{m}^2}{\text{s}}\right)^\alpha \left(\frac{\text{m}}{\text{s}}\right)^\beta \left(\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)^\omega = (kg)^{\alpha-\omega}(m)^{2\alpha+\beta+3\omega}(s)^{-(\alpha+\beta+2\omega)}$$

(a) The Planck mass is derived from the system of exponent equations: $\alpha - \omega = 1, 2\alpha + \beta + 3\omega = 0, \alpha + \beta + 2\omega = 0$, which has constraints $\alpha = -\omega$. From this, one can show that $\beta = \frac{1}{2} = -\omega$. Thus, the Planck mass is $\boxed{M_{\text{Pl}} = \sqrt{\frac{\hbar c}{G}} = 2.17 \times 10^{-8}\ \text{kg}}$

(b) To obtain units of time, we require: $\alpha - \omega = 0$, $2\alpha + \beta + 3\omega = 0$, $\alpha + \beta + 2\omega = -1$. The first equation implies that $\alpha = \omega$, and substituting into the second gives $\beta = -5\alpha$. The third equation yields $\alpha = -\frac{1}{2}$. Thus, the Planck time is $\boxed{T_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^5}} = 5.41 \times 10^{-44}\ \text{s}}$

(c) By the same procedure, we can isolate units of length through the equations $\alpha - \omega = 0, 2\alpha + \beta + 3\omega = 1, \alpha + \beta + 2\omega = 0$, giving $\alpha = \omega = \frac{1}{2}$, $\beta = -\frac{3}{2}$. Thus, the Planck length is $\boxed{L_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}\ \text{m}}$

2. If we supose that the general force law is $F = ke^2 r^\beta$ for abitrary exponent β, then the Bohr quantization conditions can be derived as follows:

$$\text{Force balancing} \implies \frac{mv^2}{r} = ke^2 r^\beta$$

$$\text{Energy conservation} \implies E = \frac{1}{2}mv^2 + V(r)$$

In this case, the potential $V(r)$ is *not* the usual Coulomb ($\frac{1}{r}$) potential, since we need to remember that $F(r) = -\nabla V(r)$. Thus, we need to substitute $V(r) = \frac{ke^2}{\beta+1} r^{\beta+1}$. Noting also from the "force" condition that $mv^2 = ke^2 r^{\beta+1}$, the energy condition becomes

$$E = \frac{1}{2}ke^2 r^{\beta+1} + \frac{ke^2}{\beta+1} r^{\beta+1} = ke^2 r^{\beta+1}\left(\frac{1}{2} + \frac{1}{\beta+1}\right)$$

We can eliminate velocity from the conservation of angular momentum condition,

$$mvr = n\hbar \quad \longrightarrow \quad v^2 = \frac{n^2\hbar^2}{m^2r^2}$$

and using the force condition, we can solve for r:

$$\frac{mv^2}{r} = ke^2r^\beta$$

$$r^{\beta+1} = \frac{mv^2}{ke^2} = \frac{mn^2\hbar^2}{m^2r^2ke^2}$$

$$r^{\beta+3} = \frac{n^2\hbar^2}{mke^2} \quad \Longrightarrow \quad r = \left(\frac{n^2\hbar^2}{mke^2}\right)^{\frac{1}{\beta+3}}$$

Substituting this back into the energy condition gives:

$$\begin{aligned} E &= k\left(\frac{n^2\hbar^2}{mke^2}\right)^{\frac{\beta+1}{\beta+3}}\left(\frac{1}{2}+\frac{1}{\beta+1}\right) \\ &= k(k)^{-\frac{\beta+1}{\beta+3}}\left(\frac{n^2\hbar^2}{me^2}\right)^{\frac{\beta+1}{\beta+3}}\left(\frac{1}{2}+\frac{1}{\beta+1}\right) \end{aligned}$$

Since $k(k)^{-\frac{\beta+1}{\beta+3}} = k^{\frac{\beta+3-\beta-1}{\beta+3}} = k^{\frac{2}{\beta+3}}$, we find that the modified Bohr energies are $\boxed{E = k^{\frac{2}{\beta+3}}\left(\frac{n^2\hbar^2}{me^2}\right)^{\frac{\beta+1}{\beta+3}}\left(\frac{1}{2}+\frac{1}{\beta+1}\right)}$

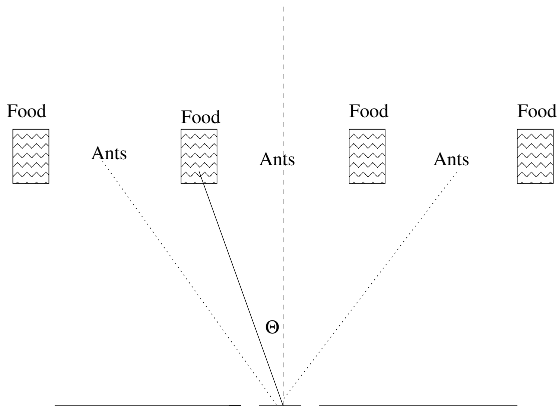
3. (a) Since the atoms in the lattice are spaced at $a = 2.15\ \mathring{A}$, we can take this to be the "slit" spacing. The peak electron signal would result at the first maximum of the multiple-slit interference pattern, which can be determined from the expression $a\sin\theta = \lambda_{dB}$, where λ_{dB} is the deBroglie wavelength of the electrons. Thus, we find $\boxed{\lambda_{dB} = 1.64\ AA = 1.64 \times 10^{-10}\ \text{m.}}$

   (b) When an electron is passed through a potential difference ΔV, it acquires an amount of energy $E = |q_e|\Delta V$, which by conservation of energy must be its kinetic energy. Note that the momentum of a particle with kinetic energy K is $p = mv = \sqrt{2mK}$, so we can find the deBroglie wavelength as follows:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_eK}} = \frac{h}{\sqrt{2m_e|q_e|\Delta V}} = 1.67 \times 10^{-10}\ \text{m}$$

   So, the two wavelengths agree! Louis deBroglie gets his Nobel Prize!

4. (a) Here, the ants walking through the two holes in the wall are like the double-slit experiment:

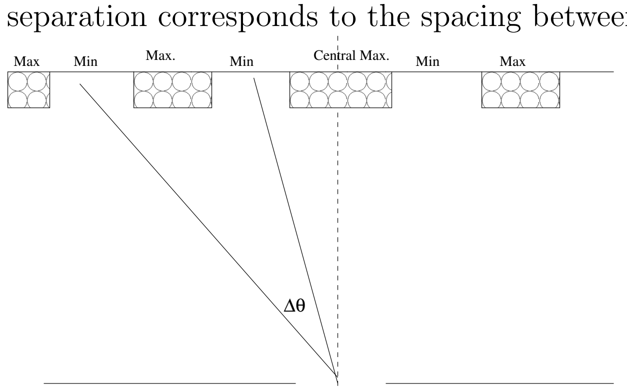


We expect lots of ants to diffract to the maxima of the interference pattern, and no ants to diffract to the minima. So, we should put the food along the minima, whose angular separation is determined by the relationship $a \sin\theta = \left(n + \frac{1}{2}\right)\lambda_{dB}$, where m is an integer representing the minimum. In this case, we're interested in the first minimum, so $n = 0$. Thus, keeping in mind that $\lambda_{dB} = \frac{h}{mv}$, the food should be placed at the angular separation θ described by:

$$\theta = \frac{\lambda_{dB}}{2a} = \frac{h}{2am_{\text{ant}}v} = \frac{10^{-6}}{2(10^{-4})(0.5)(0.1)} = 0.1 \text{ radians}$$

The food will be safe at an angular distance of $\boxed{\theta = 0.1 \text{ rad} \approx 5.7^\circ}$ from the central pile of ants (maximum).

(b) For people to diffract in a single-slit experiment, we can assume that their separation corresponds to the spacing between each successive minimum:



For small angles, we have $\theta_n \approx \frac{n\lambda}{D}$, so between two successive angles θ_n and θ_{n+1}, we find

$$\Delta\theta = \theta_{n+1} - \theta_n = \frac{(n+1)\lambda}{D} - \frac{n\lambda}{D} = \frac{\lambda}{D}$$

If the people are 1 m apart on the wall (5 m from the door), the angular separation is $\Delta\theta = \frac{1}{5} = 0.2$ rad. So, we have

$$\Delta\theta = \frac{h}{mvD}$$

and we can solve for Planck's Constant accordingly:
$\boxed{h = mvD\Delta\theta = 9.0 \text{ J}\cdot\text{s.}}$ That is, Planck's Constant would have to be about 34 orders of magnitude bigger than it is for quantum effects to become apparent on such a large scale.