A SEQUENCE OF HERMITE INTERPOLATING-LIKE POLYNOMIALS FOR APPROXIMATING ARCTANGENT

HERBERT A. MEDINA

1. Introduction

The Taylor series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1}$$

was discovered by the Scotsman James Gregory in 1671 ([B, Ch. 12]). It is not hard to show that the series converges uniformly to $\arctan x$ on $[-1,1]$; thus, the series produces the following sequence of Taylor polynomials in $\mathbb{Q}[x]$ that converges uniformly to $\arctan x$ on $[-1,1]$:

$$T_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\, x^{2k+1}.$$

Like the Taylor polynomials for several other classical functions, e.g., $\cos x$, $\sin x$, and e^x, this sequence of polynomials is very easy to describe and work with; but unlike those Taylor polynomials with factorials in the denominators of their coefficients, it does not converge rapidly for all "important" values of x. In particular, it converges extremely slowly to $\arctan x$ when $|x|$ is near 1. For example, if $x = 0.95$, we would need to use T_{28}, a polynomial of degree 57, to get three decimal places of accuracy for $\arctan(0.95)$; if $x = 1$, we would need to use T_{500}, a polynomial of degree 1001, to get three decimal places of accuracy for $\arctan 1$.

Indeed, for $x \in [0,1]$

$$\begin{aligned}
\arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\
&= \int_0^x \sum_{k=0}^{\infty} (-1)^k\, t^{2k}\,dt \\
&= \int_0^x \sum_{k=0}^{n} (-1)^k\, t^{2k} + \sum_{k=n+1}^{\infty} (-1)^k\, t^{2k}\,dt \\
&= T_n(x) + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt;
\end{aligned}$$

therefore, $|\arctan x - T_n(x)| = \displaystyle\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \geq \int_0^x \frac{t^{2n+2}}{2}\,dt = \frac{x^{2n+3}}{2(2n+3)}$. Thus, as $x \to 1$, $T_n(x)$ cannot approximate $\arctan x$ any better than $\frac{1}{2(2n+3)} = \frac{1}{2(\text{degree } T_n)+4}$. The same holds true for x values near -1. It is only fair to note that the sequence $\{T_n(x)\}$ converges to $\arctan x$ reasonably fast for x values near 0.

The purpose of this article is to present another very elementary, easily-described sequence of polynomials in $\mathbb{Q}[x]$ that approximates $\arctan x$ uniformly on $[0,1]$ and which does so much more rapidly than the sequence of Taylor polynomials centered at 0. We note that such an approximating sequence provides, via the identities $\arctan x = -\arctan(-x) = \frac{\pi}{2} - \arctan(\frac{1}{x})$, a method of approximating $\arctan x$ for all $x \in \mathbb{R}$. The approximating sequence arises from the simple family of rational functions $\left\{\dfrac{x^{4m}(1-x)^{4m}}{1+x^2}\right\}_{m\in\mathbb{N}}$.

2. THE SEQUENCE OF POLYNOMIALS AND ITS RATE OF CONVERGENCE

We begin with an algebraic computation.

Lemma 1. Define $p_1(x) = 4 - 4x^2 + 5x^4 - 4x^5 + x^6$ and $p_m(x) = x^4(1-x)^4 p_{m-1}(x) + (-4)^{m-1}p_1(x)$ for $m \geq 2$. Then

$$\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) + \frac{(-4)^m}{1+x^2}, \text{ for all } m \in \mathbb{N}.$$

Proof. We argue by induction on m. A computation shows that $\frac{x^4(1-x)^4}{1+x^2} = p_1(x) - \frac{4}{1+x^2}$. Assume $\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) + \frac{(-4)^m}{1+x^2}$. Now

$$\begin{aligned}
\frac{x^{4(m+1)}\ (1-x)^{4(m+1)}}{1+x^2} &= x^4(1-x)^4\Big(p_m(x) + \frac{(-4)^m}{1+x^2}\Big) \\
&= x^4(1-x)^4 p_m(x) + (-4)^m \frac{x^4(1-x)^4}{1+x^2} \\
&= x^4(1-x)^4 p_m(x) + (-4)^m \Big(p_1(x) - \frac{4}{1+x^2}\Big) \\
&= x^4(1-x)^4 p_m(x) + (-4)^m p_1(x) + \frac{(-4)^{m+1}}{1+x^2} \\
&= p_{m+1}(x) + \frac{(-4)^{m+1}}{1+x^2}.
\end{aligned}$$

□

A calculus computation shows that $x(1-x) \leq \frac{1}{4}$ on $[0,1]$. Thus, $\dfrac{x^{4m}(1-x)^{4m}}{1+x^2} \leq \Big(\dfrac{1}{4}\Big)^{4m}$ on $[0,1]$, and

$$\int_0^x \frac{t^{4m}(1-t)^{4m}}{1+t^2}\, dt \leq \Big(\frac{1}{4}\Big)^{4m} x \leq \Big(\frac{1}{4}\Big)^{4m}, \ \forall x \in [0,1].$$

The result of the lemma can be rewritten as

$$\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) - \frac{(-1)^{m+1}4^m}{1+x^2}.$$

Thus,

$$\Big|\int_0^x p_m(t) - \frac{(-1)^{m+1}4^m}{1+t^2}\, dt\Big| \leq \Big(\frac{1}{4}\Big)^{4m}.$$

Dividing by $(-1)^{m+1}4^m$ and integrating the second term on the left we get

$$\Big|\int_0^x \frac{(-1)^{m+1}}{4^m} p_m(t)\,dt - \arctan x\Big| \le \Big(\frac{1}{4}\Big)^{5m}. \tag{1}$$

So

$$h_m(x) = \int_0^x \frac{(-1)^{m+1}}{4^m} p_m(t)\,dt \tag{2}$$

defines a sequence of polynomials in $\mathbb{Q}[x]$ which converges uniformly on $[0,1]$ to $\arctan x$ at a much faster rate than the sequence of Taylor polynomials.

To get a better sense of the rate of convergence, we notice that the degree of p_m is $8m-2$, and hence h_m has degree $8m-1$. We also note that $4^{5m} = \big(4^{5/8}\big)^{8m} = \big(4^{5/8}\big)^{8m-1+1}$; we use this form in (1), and summarize our results in the following theorem.

Theorem 1. For $m \in \mathbb{N}$, define $p_m(t)$ as in Lemma 1 and $h_m(x)$ as in (2). Then

$$\big|h_m(x) - \arctan x\big| \le \Big(\frac{1}{4^{5/8}}\Big)^{\text{degree } h_m + 1} \quad \text{for all } x \in [0,1]. \tag{3}$$

3. Examples and observations about the approximating sequence

We begin this section with some concrete examples. Evaluating

$$h_2(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{5\,x^9}{48} + \frac{x^{10}}{20} - \frac{43\,x^{11}}{176} + \frac{x^{12}}{4} - \frac{27\,x^{13}}{208} + \frac{x^{14}}{28} - \frac{x^{15}}{240}$$

at $x = 0.95$ and $x = 1$ we find that at both points, the approximation to $\arctan x$ is within 2.28×10^{-7}, better than six decimal places of accuracy with a polynomial of much smaller degree than the Taylor polynomials (which gave much less accuracy) that we looked at in the Introduction. If we consider

$$\begin{aligned} h_7(x) \;=\; & x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{15}}{15} + \frac{x^{17}}{17} - \frac{x^{19}}{19} + \frac{x^{21}}{21} - \frac{x^{23}}{23} + \frac{x^{25}}{25} \\ & - \frac{x^{27}}{27} + \frac{565\,x^{29}}{16384} - \frac{7\,x^{30}}{122880} - \frac{16007\,x^{31}}{507904} - \frac{203\,x^{32}}{32768} + \frac{18241\,x^{33}}{270336} - \frac{11879\,x^{34}}{69632} \\ & + \frac{170129\,x^{35}}{286720} - \frac{68063\,x^{36}}{36864} + \frac{2767847\,x^{37}}{606208} - \frac{1454473\,x^{38}}{155648} + \frac{10355263\,x^{39}}{638976} \\ & - \frac{489259\,x^{40}}{20480} + \frac{5016623\,x^{41}}{167936} - \frac{1361617\,x^{42}}{43008} + \frac{5012527\,x^{43}}{176128} - \frac{489259\,x^{44}}{22528} \\ & + \frac{10371647\,x^{45}}{737280} - \frac{1454473\,x^{46}}{188416} + \frac{2751463\,x^{47}}{770048} - \frac{68063\,x^{48}}{49152} + \frac{178321\,x^{49}}{401408} \\ & - \frac{11879\,x^{50}}{102400} + \frac{10049\,x^{51}}{417792} - \frac{203\,x^{52}}{53248} + \frac{377\,x^{53}}{868352} - \frac{7\,x^{54}}{221184} + \frac{x^{55}}{901120}, \end{aligned}$$

a polynomial of smaller degree than the Taylor polynomials considered in the Introduction, (3) guarantees that the approximation on $[0,1]$ is accurate to within 8.47×10^{-22}. Thus, $4\,h_7(1) = \frac{50611943354106452425449}{161102819285860855603200}$ accurately gives the first 20 digits to the right of the decimal point for π. (All symbolic and numerical computations were carried out using the computer algebra system *Mathematica* 4.1.)

Another attractive feature about the sequence $\{h_m\}$ is that the error levels off as $x \to 1$. Indeed, using basic calculus, one can prove that the general shape of the error curve is that given in Figure 1. The shape of the error curve of the Taylor polynomial approximation is given in Figure 2 and shows that the Taylor polynomial approximation experiences much larger deficiencies as $x \to 1$. We also note that, like the Taylor polynomials, the h_m provide one sided approximations to $\arctan x$. Indeed, it is not hard to see that $h_m(x) - \arctan x$ is positive when m is odd and negative when m is even.

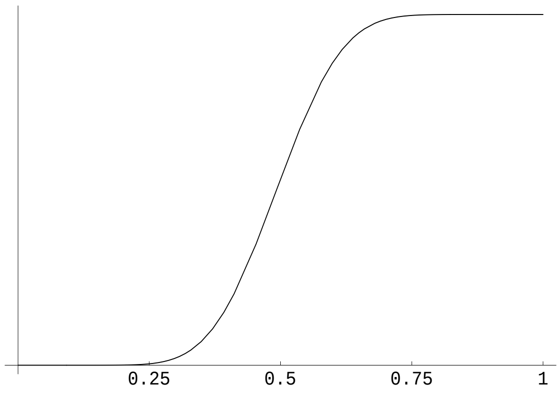

FIGURE 1. Plot of $|h_m(x) - \arctan x|$.

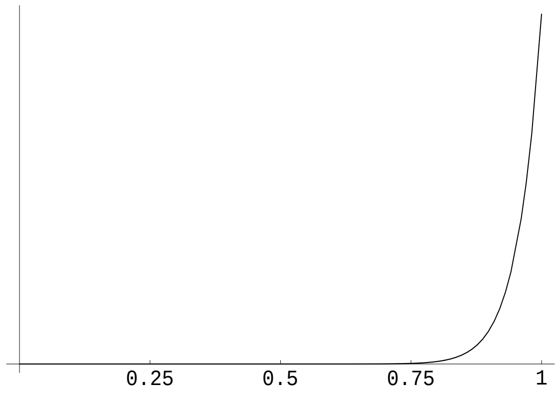

FIGURE 2. Plot of $|T_{10}(x) - \arctan x|$. Error curves for other T_n are similar.

The Taylor polynomials for $\arctan x$ are constructed by matching the derivatives of $\arctan x$ at $x = 0$. A generalization of Taylor polynomials is the following.

Definition. Let $x_0, x_1, \ldots, x_n$ be distinct points in $[a, b]$ and m_i a nonnegative integer associated with x_i for $i = 0, 1, \ldots, n$. Suppose that $f \in C^m[a, b]$ where $m = \max_{0 \le i \le n} m_i$. The *Hermite Interpolating*[1] polynomial approximating f is the polynomial $P(x)$ of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k} \text{ for each } i = 0, 1, \ldots, n \text{ and } k = 0, 1, \ldots, m_i.$$

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[1] Also referred to as *osculating* polynomial.

Basic algorithms for computing Hermite interpolating polynomials are well known and can be found in many numerical analysis texts (e.g., [BF]). We note that if we have $x_i = 1$ in the definition above, then the Hermite-interpolating polynomial for $\arctan x$ cannot be in $\mathbb{Q}[x]$ because $\arctan 1 = \frac{\pi}{4}$. Thus, none of the h_m is a Hermite-interpolating polynomial. Nevertheless, the next theorem shows that, surprisingly, the h_m are similar to Hermite-interpolating polynomials as they match derivatives of $\arctan x$ at both $x = 0$ and $x = 1$.

Theorem 2. For any $m \geq 1$, $h_m^{(n)}(0) = \arctan^{(n)}(0)$ and $h_m^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$. Moreover, if $g(x)$ is a polynomial of degree $8m$ such that $g(0) = \arctan 0$, $g^{(n)}(0) = \arctan^{(n)}(0)$ and $g^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$, then $g = h_m$.

Proof. We deal with the $x = 1$ case first. Use (2) and Lemma 1 to note that

$$\begin{aligned} h_m'(x) &= \frac{(-1)^{m+1}}{4^m} p_m(x) = \frac{(-1)^{m+1}}{4^m}\left(\frac{x^{4m}(1-x)^{4m}}{1+x^2} - \frac{(-4)^m}{1+x^2}\right) \\ &= \left(\frac{(-1)^{m+1}}{4^m}\frac{x^{4m}}{1+x^2}\right)(1-x)^{4m} + \frac{1}{1+x^2}. \end{aligned} \tag{4}$$

Using $\arctan' x = \frac{1}{1+x^2}$ on the second term and the product rule for differentiation, $\Big(f(x)g(x)\Big)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x)$, on the first, we get

$$h_m^{(n)}(x) = \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{(-1)^{m+1}}{4^m}\frac{x^{4m}}{1+x^2}\right)^{(n-1-k)}\Big((1-x)^{4m}\Big)^{(k)} + \arctan^{(n)}(x).$$

For $0 \leq k \leq n-1$, $\Big((1-x)^{4m}\Big)^{(k)}\Big|_{x=1} = (-1)^k(4m)(4m-1)\cdots(4m-k)(1-x)^{4m-k}\Big|_{x=1} = 0$. Thus, $h_m^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$.

To prove the assertion at $x = 0$, we can rewrite the first summand in (4) as $\left(\frac{(-1)^{m+1}}{4^m}\frac{(1-x)^{4m}}{1+x^2}\right)x^{4m}$, and follow the same steps used above.

If g is a polynomial with the properties stated, then $g - h_m$ is a polynomial of degree $8m$ whose first $4m$ coefficients must be 0, because $g^{(n)}(0) - h_m^{(n)}(0) = 0$ for $0 \leq n \leq 4m$. Hence $(g - h_m)(x) = x^{4m+1}q(x)$ where q is a polynomial of degree $4m-1$; write $q(x) = \sum_{k=0}^{4m-1} a_k(x-1)^k$. Inductive use of the product rule to compute $(g - h_m)^{(k)}(1)$ shows that $a_k = C_k a_0$ for $1 \leq k \leq 4m-1$ where $C_k \neq 0$; therefore its use on $(g - h_m)^{(4m)}$ shows $(g - h_m)^{(4m)}(1) = Ca_0$ where $C \neq 0$. Thus $a_0 = 0$ and $a_k = 0$ for $1 \leq k \leq 4m-1$. □

4. CLOSED-FORM FORMULAS FOR THE POLYNOMIALS

The reader will notice that the first half of h_2 and h_7 consist of terms in the Taylor series for $\arctan x$. We prove that this is the case in general and give a closed-form formula for h_m in the following proposition.

Proposition 1. For any $m \in \mathbb{N}$,

$$p_m(t) = (-1)^{m+1}4^m \sum_{k=0}^{2m-1}(-1)^k t^{2k} + t^{4m}(5-4t+t^2)\sum_{k=1}^{m}(1-t)^{4(m-k)}(-4)^{k-1}, \quad (5)$$

and hence

$$h_m(x) = T_{2m-1}(x) + \frac{(-1)^{m+1}}{4^m}\int_0^x t^{4m}(5-4t+t^2)\sum_{k=1}^{m}(1-t)^{4(m-k)}(-4)^{k-1}\,dt. \quad (6)$$

Proof. The second assertion follows directly from the first. To establish the first, we begin with the result of Lemma 1, $p_m(t) = \frac{t^{4m}(1-t)^{4m}}{1+t^2} - \frac{(-4)^m}{1+t^2}$ and recall that $\frac{1}{1+t^2} = \sum_{k=0}^{2m-1}(-1)^k t^{2k} + \frac{t^{4m}}{1+t^2}$. Thus,

$$\begin{aligned} p_m(t) &= \frac{t^{4m}(1-t)^{4m}}{1+t^2} - (-4)^m\Big(\sum_{k=0}^{2m-1}(-1)^k t^{2k} + \frac{t^{4m}}{1+t^2}\Big) \\ &= -(-4)^m \sum_{k=0}^{2m-1}(-1)^k t^{2k} + \frac{t^{4m}(1-t)^{4m} - (-4)^m t^{4m}}{1+t^2} \\ &= (-1)^{m+1}4^m \sum_{k=0}^{2m-1}(-1)^k t^{2k} + t^{4m}\Big(\frac{(1-t)^{4m} - (-4)^m}{1+t^2}\Big). \qquad (7) \end{aligned}$$

We use the algebraic identity $y^n - z^n = (y-z)\sum_{k=1}^{n-1} y^{n-k}z^{k-1}$ to conclude that $(1-t)^{4m} - (-4)^m = \big((1-t)^4\big)^m - (-4)^m = \big((1-t)^4 + 4\big)\sum_{k=1}^{m-1}(1-t)^{4(m-k)}(-4)^{k-1}$. A computation shows that $(1-t)^4 + 4 = (1+t^2)(5-4t+t^2)$. The result now follows from (7). □

An alternative closed form formula for p_m, and hence for the approximating sequence, is given by the following proposition; we leave the proof (which is independent of the previous proposition) to the reader.

Proposition 2. For $m \geq 2$,

$$p_m(t) = p_1(t)\ \sum_{k=0}^{m-1}(-4)^{m-1-k}t^{4k}(1-t)^{4k}.$$

Both of the closed form formulas presented thus far to not provide explicit formulas for the coefficients of h_m. The first two parts of the next lemma are key in establishing formulas for the coefficients; the third part is an interesting symmetry property of the coefficients.

Lemma 2. For $m \in \mathbb{N}$, write $\dfrac{(1-t)^{4m}}{1+t^2} = \sum_{j=0}^{4m-2} a_j\, t^j + \dfrac{r_m(t)}{1+t^2}$, where r_m is a polynomial with $\deg(r_m) < 2$.

(i.) $r_m(t) = (-1)^m 4^m$.

(ii.) $a_{2j} = (-1)^{j+1} \sum_{k=j+1}^{2m} \binom{4m}{2k} (-1)^k$ and $a_{2j-1} = (-1)^{j+1} \sum_{k=j}^{2m-1} \binom{4m}{2k+1} (-1)^k$.

(iii.) For $l = 1, \ldots, m$, $a_{4m-2l} - a_{2(l-1)} = (-1)^{m+l-1} 4^m$, $a_{4m-(2l+1)} - a_{2l-1} = 0$. (e.g.,

$a_{4m-2} - a_0 = (-1)^m 4^m$, $a_{4m-3} - a_1 = 0$, $a_{4m-4} - a_2 = (-1)^{m+1} 4^m, \ldots$.)

Proof. Write

$$\begin{aligned}\frac{(1-t)^{4m}}{1+t^2} &= \frac{\sum_{k=0}^{4m} \binom{4m}{k} (-1)^k t^k}{(1+t)^2} \\ &= \underbrace{\sum_{k=0}^{2m} \binom{4m}{2k} \frac{t^{2k}}{1+t^2} - \sum_{k=1}^{2m-1} \binom{4m}{2k+1} \frac{t^{2k+1}}{1+t^2}}_{S_E - S_O.}\end{aligned}$$

Using $\frac{t^{2k}}{1+t^2} = (-1)^{k+1} \sum_{j=1}^{k} (-1)^{j+1} t^{2(j-1)} + \frac{(-1)^k}{1+t^2}$, for $k \geq 1$, we write

$$S_E = \frac{1}{1+t^2} + \sum_{k=1}^{2m} \binom{4m}{2k} \left((-1)^{k+1} \sum_{j=1}^{k} (-1)^{j+1}\, t^{2(j-1)} + \frac{(-1)^k}{1+t^2} \right).$$

We collect the polynomial and non-polynomial parts

$$S_E = \sum_{k=1}^{2m} \binom{4m}{2k} \left((-1)^k \sum_{j=1}^{k} (-1)^j\, t^{2(j-1)} \right) + \sum_{k=0}^{2m} \binom{4m}{2k} \frac{(-1)^k}{1+t^2}.$$

Using the identity $\sum_{k=0}^{2m} \binom{4m}{2k} (-1)^k = (-1)^m 4^m$, the non-polynomial part becomes $\frac{(-1)^m 4^m}{1+t^2}$. (An easy way to arrive at the identity is to expand $(1-i)^{4m}$, notice that it equals $(-1)^m 4^m$, and equate real and imaginary parts. The expansion also gives $\sum_{k=0}^{2m-1} \binom{4m}{2k+1} (-1)^k = 0$; we will use this later in the argument.) We change the order of summation on the polynomial part to get

$$\sum_{j=1}^{2m} \left((-1)^j \sum_{k=j}^{2m} \binom{4m}{2k} (-1)^k \right) t^{2(j-1)}.$$

A similar procedure as that done on S_E shows that

$$S_O = \sum_{j=1}^{2m-1} \left((-1)^j \sum_{k=j}^{2m-1} (-1)^k \binom{4m}{2k+1} \right) t^{2j-1} + \frac{t}{1+t^2} \sum_{k=0}^{2m-1} \binom{4m}{2k+1} (-1)^k.$$

The parenthetical observation earlier in the argument, immediately gives that the second summand is zero. We have established that

$$\begin{aligned} \frac{(1-t)^{4m}}{1+t^2} &= \sum_{j=1}^{2m} \left((-1)^j \sum_{k=j}^{2m} \binom{4m}{2k} (-1)^k \right) t^{2(j-1)} + \frac{(-1)^m 4^m}{1+t^2} \\ &+ \sum_{j=1}^{2m-1} \left((-1)^{j+1} \sum_{k=j}^{2m-1} \binom{4m}{2k+1} (-1)^k \right) t^{2j-1}, \end{aligned} \tag{8}$$

and this proves first two parts of the lemma.

For $1 \leq l \leq m$,

$$a_{4m-2l} - a_{2(l-1)} = (-1)^{2m-l+1} \sum_{k=2m-l+1}^{2m} \binom{4m}{2k} (-1)^k - (-1)^l \sum_{k=l}^{2m} \binom{4m}{2k} (-1)^k.$$

We note that $\binom{4m}{2k} = \binom{4m}{4m-2k} = \binom{4m}{2(2m-k)}$ and let $i = 2m - k$ to get

$$\begin{aligned} a_{4m-2l} - a_{2(l-1)} &= (-1)^{l-1} \sum_{k=2m-l+1}^{2m} \binom{4m}{2k} (-1)^k + (-1)^{l-1} \sum_{i=0}^{2m-l} \binom{4m}{2i} (-1)^i \\ &= (-1)^{l-1} \sum_{k=0}^{2m} \binom{4m}{2k} (-1)^k \\ &= (-1)^{l-1} (-1)^m 4^m. \end{aligned}$$

This proves the even part of (iii); the odd part follows from a similar computation. □

Combining the results of Lemmas 1 and 2, we see that

$$p_m(t) = (-1)^m 4^m \sum_{j=1}^{2m} (-1)^j \, t^{2(j-1)} + \sum_{j=0}^{4m-2} a_j \, t^{4m+j}, \tag{9}$$

where the a_j are those given in Lemma 2. We now can give a closed form formula for the coefficients of the approximating polynomials.

Theorem 3. For $m \geq 1$,

$$h_m(x) = \sum_{j=1}^{2m} \frac{(-1)^{j+1}}{2j-1} x^{2j-1} + \sum_{j=0}^{4m-2} \frac{a_j}{(-1)^{m+1}4^m(4m+j+1)} x^{4m+j+1},$$

where $a_{2i} = (-1)^{i+1} \sum_{k=i+1}^{2m} \binom{4m}{2k}(-1)^k$ and $a_{2i-1} = (-1)^{i+1} \sum_{k=i}^{2m-1} \binom{4m}{2k+1}(-1)^k$.

Proof. The result follows from (9) and (2). □

5. Concluding Remarks and Questions

How does $\{h_m\}$ compare to a state-of-the-art, multi-precision numerical package for approximating $\arctan x$? A comparison was made in approximating, requiring fifty-digits of accuracy, arctan at 2000, fifty-digit, pseudo-random numbers in $[0.9, 1]$ using a FORTRAN version of $h_{17}(x)$ and David Smith's multi-precision FORTRAN numerical package [S1] on a G3 Macintosh computer; the results are as follows:

- $h_{17}(x)$ did the computation in 3.17 seconds.
- Smith's numerical package did the computation in 1.83 seconds. (Smith's package uses a Newton iteration with increasing variable precision to solve $\tan y = x$ for approximating $\arctan x$ when x is near 1; $\tan y$ is evaluated as $\frac{\sin y}{\sqrt{1-\sin^2 y}}$ using techniques explained in [S2] to speed up the evaluation of $\sin y$; the initial guess for the iteration comes from the built-in, double-precision arctan function. The Taylor series is used when x is very close to 0.)

The deficiency in using the h_m in a numerical package to approximate arctan is that the error bound does not have any factorials in the denominator. The benefits are that the polynomials provide a straight forward approach, are easy to compute and give an actual function approximation of this transcendental function that is not that much slower than a sophisticated numerical package.

Indeed, because of Theorem 3, the sequence $\{h_m\}$ is easily programmable and gives functions that will approximate arctan to a prescribed accuracy for $x \in [0,1]$. Appendix A contains a *Mathematica* program that computes the necessary $h_m(x)$ to approximate $\arctan x,\ x \in [0,1]$, to "`numberofdigits`" decimal places of accuracy. The program computes the h_m much faster than the direct computation of the coefficients, via solving a system of linear equations, suggested by the result of Theorem 2. For example, to find a polynomial h_m of degree $8m$ such that $h_m(0) = \arctan 0$, $h_m^{(n)}(0) = \arctan^{(n)}(0)$ and $h_m^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$, we solved for the coefficients of h_m by solving the system of linear equations (the $8m+1$ coefficients being the unknown variables), generated by the conditions on h_m. For $m = 5,\ 10,\ 15$, and 20, on a 733 MHz Pentium III computer running *Mathematica* 4.1, this method yielded computation times of $0.82,\ 13.63,\ 88.1$ and 332.08 seconds respectively. The program above (without the "Print" statements) computed the h_m in $0.0,\ 0.06,\ 0.11$ and 0.11 seconds respectively.

The keys to the approximating sequence $\{h_m\}$ are that the family of polynomials $x^{4m}(1-x)^{4m},\ m \in \mathbb{N}$ leaves an integer remainder when divided by $1+x^2$ and that the members of the family are very small for $x \in [0,1]$. There are other families of polynomials with this property. Is there another simple one that gives a faster approximation to $\arctan x$? In the best case, is there one with a closed form and with the desirable factorials in the denominator of the error bound?

The results herein were stumbled upon after the author became intrigued by and curious about $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$; that is, $4(h_1(1) - \arctan 1) = \frac{22}{7} - \pi$. Is there a simple closed-form formula for $4h_m(1)$? If so, it would provide a sequence of rationals for approximating π. Another, probably-very-difficult, problem is to find (determine whether there exists) an easily-describable sequence of polynomials $\{g_n\}$ such that $4g_n(1)$ is always a convergent in the continued fraction expansion of π.

APPENDIX A: *Mathematica* PROGRAM TO COMPUTE h_m

```
Clear[h, m, a, numberofdigits]
numberofdigits=20;
m=Floor[-(1/5) Log[4,5*0.1^(numberofdigits+1)]]+1;
Do[a[2j]=(-1)^(j+1) Sum[Binomial[4m,2k] (-1)^k, {k,j+1,2m}];
   a[2j-1]=(-1)^(j+1) Sum[Binomial[4m,2k+1] (-1)^k, {k,j,2m-1}],
   {j,0,2m-1}];
h[m,x_]:=Sum[(-1)^(j+1) / (2j-1) x^(2j-1), {j,1,2m}] +
         Sum[a[j]/((-1)^(m+1) 4^m (4m+j+1)) x^(4m+j+1), {j,0, 4m-2}];
Print["h[",m,",x] given below computes ArcTan[x] with ",
      numberofdigits," digits of accuracy for x in [0,1]."]
Print["h[m,x] = ", h[m,x]]
```

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DEPARTMENT OF MATHEMATICS, LOYOLA MARYMOUNT UNIVERSITY, LOS ANGELES, CA 90045

E-mail address: `hmedina@lmu.edu`