Vector or Pseudovector?

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By using a corner reflector it is possible to perform an inversion or improper transformation thereby identifying a physical quantity as a scalar, pseudoscalar, vector or pseudovector. This visualization can aid students when they are first learning about vector transformations. An understanding of symmetry and transformation properties can also be beneficial when one studies electromagnetic and weak interactions.

Introduction

Often in the early chapters of a mechanics or electricity and magnetism text, there is a comment or footnote describing pseudovectors. While talking about inversion transformations of various quantities it will be pointed out that not all “vectors” actually meet all of the criteria to be a vector. These comments will often generate questions from students, many of whom struggle to visualize the transformation. This simple demonstration shows the student how easily understood quantities such as displacement, velocity and angular velocity transform though the use of three orthogonal mirrors.

The corner reflector will invert any quantity via three reflections. Each mirror serves to “reverse” the component of the quantity which is perpendicular to that mirror. So long as one views the image which has been reflected from all three mirrors, rather than one of the other images, the quantity will have been inverted. This process is also referred to as an improper or parity transformation.

In addition to providing the student with a better understanding of the mathematical formulism of vector algebra, it also gives the student an introduction in symmetry. Emmy Noether proved that if a system contained symmetry then there would be physical quantities that would be conserved. For example, when a system contains rotation symmetry, angular momentum is conserved. When one attempts to conjecture what the properties of a magnetic monopole might be, an understanding of spatial inversion, as well as time reversal and charge conjugation, can be beneficial. Parity invariance also played an important role in the understanding of the weak interaction.

Demonstration

Using three 12” x 12” mirror tiles that are readily available from hardware and home décor stores, a corner reflector was constructed. By using these larger tiles, instead of the smaller corner reflectors one often finds in optical demonstrations, it makes it easier for
students to see the transformed quantity as the reflected image is viewable over a larger angle. The tiles were fastened together using mirror caulk and aluminum corner braces.

A velocity vector, as represented by an arrow, and angular velocity are shown in the first illustration. For this still photo, a curved arrow was attached to the spinning disc to illustrate direction of rotation. Included next to the two quantities is the basis of a right-handed (Cartesian) coordinate system. According to this coordinate system, the velocity vector is in the positive y direction, and the angular velocity is in the positive x direction.

From the reflected image, we see that the transformed velocity vector is now directed in the negative y direction, indicating that it is a true vector. That is, all of the components reverse direction when inverted. However, when we use the right-hand rule on the disc’s reflection, we see that the angular velocity is still pointing in the positive x direction. This is due to the fact that cross products, such as \( \mathbf{r} \times \mathbf{v} \), are pseudovectors. When pseudovectors are transformed via an inversion, they continue to point the same direction. We can now see that not every “vector” is really a vector.

Of course, the transformation properties of physical quantities besides velocity could be examined. Momentum, force, and current density are just a few vectors and quantities such as angular momentum, torque, and magnetic moment are pseudovectors. The fact that many of the pseudovectors are related to rotation motion gives rise to another name for these quantities - axial vectors.

Vector fields can also be classified. If one imagines a uniform electric field between two conducting plates, this will be directed the other way when it is transformed, just as velocity or displacement, indicating that the electric field is a vector. By contrast we can consider a magnetic field due to an infinitely long current carrying wire that passes through the center of the disc in figure 1. For a current that is directed toward from the camera, the magnetic field will have the same orientation as the angular velocity arrow. We see that the magnetic field is another example of a pseudovector.

At first glance one might think that the infinite wire example provides a paradox as the magnetic field transforms as a pseudovector, but the current, like the velocity vector, is a true vector. In fact, all of electromagnetism is invariant under inversion. One only has to look to the Biot-Savart law to see through this apparent paradox. The magnetic field depends on the cross product of the length of the current carrying wire and the unit vector that points to the location at which the field is to be determined. Just as with the angular velocity, the cross product of two vectors is a pseudovector.

In fact, one must always equate vectors to vectors and pseudovectors to pseudovectors. This fact can assist students when they are grappling with a situation where the exact formulation is not known. For example, when determining the magnetic field due to an infinite sheet of current, it is known that the magnetic field must depend on pseudovectors, rather than vectors. This implies that the magnetic field depends on the cross product of the current density and normal vector, rather than either vector individually.
In addition to showing the difference between vectors and pseudovector, the corner reflector can show how scalars differ from pseudoscalar. For the scalar, we consider the dot product of vectors A and B in figure 2. If one looks at the transformed vectors, one can see that the dot product remains a positive quantity as the two vectors have the same relative orientation. Figure 3 shows three vectors which can be combined in the triple scalar product \((A \times B) \cdot C\), which is a positive quantity. For the transformed product we first determine the direction of the inverted \(A \times B\), then dot this with the transformed vector A. We see that the transformed product is actually negative as A is pointing in the opposite direction of \(A \times B\), illustrating that \((A \times B) \cdot C\) is not a scalar, but rather a pseudoscalar.

**Summary**

With this simple demonstration, students have the chance to see how various quantities differ when they undergo an inversion. Not only does this serve as an initiation into the meaning of vectors and pseudovectors, it also can be an opportunity for a student to begin to think about parity, time reversal\(^{10}\), charge conjugation and other symmetry related topics.

**Bibliography**

10. Computer animations and movies can be easily viewed backwards using Apple's Quicktime Player. Once the file is opened within Quicktime, simply hold down the shift key while double-clicking on the movie image. This allows one to demonstrate time reversal without having to edit the animation or movie.
Figures

1. A corner reflector is used to invert a vector and pseudovector. The arrow, which points in the positive y-direction, produces a reflection which points in the opposite direction, indicating that it is a vector. Conversely, the spinning disk, which has an angular velocity in the positive x-direction, produces a reflection that has the same rotation. Quantities such as this which do not change signs after an inversion are known as pseudovectors.

2. The dot product of the two vectors shown in the photo is a positive quantity, as is the dot product of the two reflected vectors. This indicates that the dot product is a scalar.
3. This photo shows how the triple scalar product, \((A \times B) \cdot C\), transforms under an inversion. \(A \times B\) and \(C\) are both directed upward, resulting in a positive quantity. However, while the reflected image of \(A \times B\) is directed also directed in the positive \(z\)-direction, the reflection of \(C\) is directed downward. The fact that the reflection of the triple scalar product is a different sign than the original, shows that it is a pseudoscalar.