Math 496
Solutions to Homework

1. Show that $\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}$ is a rational number. Hint: Find rational numbers a, b, c, and d such that $(a+b\sqrt{3})^3 = 26+15\sqrt{3}$ and $(c+d\sqrt{3})^3 = 26-15\sqrt{3}$.

   Following the hint, we see that if $(a+b\sqrt{3})^3 = 26-15\sqrt{3}$, then

   $$\begin{aligned} a^3+9ab^2 &= 26 \qquad \text{and} \\ 3a^2b+3b^3 &= 15. \end{aligned}$$

   Checking to see that $a=2$, $b=1$ is a solution to this equation, we then have that $(2+\sqrt{3})^3 = 26+15\sqrt{3}$, and hence

   $$\sqrt[3]{26+15\sqrt{3}} = 2+\sqrt{3}.$$

   Similarly, one checks that $(2-\sqrt{3})^3 = 26-15\sqrt{3}$ so that

   $$\sqrt[3]{26-15\sqrt{3}} = 2-\sqrt{3}.$$

   Consequently, $\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}} = 4$ which is a rational number.

2. Find a polynomial with rational coefficients that $\sqrt[3]{26+15\sqrt{3}}$ is a root of. For this problem, there are a variety of answers. By the previous problem, $\sqrt[3]{26+15\sqrt{3}} = 2+\sqrt{3}$. If we write $a = 2+\sqrt{3}$, then we know that $a-2=\sqrt{3}$ so that $(a-2)^2 = 3$. Thus

   $$a^2-4a+4 = 3,$$

   yielding that $a^2-4a+1=0$. Thus a is a root of x^2-4x+1. To check this answer, we can use the quadratic formula to see that indeed, a is one of the roots of this equation.

   A second solution comes by writing $a^3 = 26-15\sqrt{3}$, so that we obtain $a^3-26 = 15\sqrt{3}$, and squaring both sides we obtain that $a^6-52a^3+676 = 675$. Consequently, $a^6-52a^3+1=0$, and a is a root of x^6-52x^3+1.

Is there a relationship between our two answers? Of course, there should be since we know from abstract algebra that the minimum polynomial for a should divide every other polynomial that has a as a root. Doing long division, we see that

$$x^6 - 52x^3 + 1 = (x^2 - 4x + 1)(x^4 + 4x^3 + 15x^2 + 4x + 1).$$

Other students came up with a solution by looking at $a^2 = 7 + 4\sqrt{3}$, and subtracting off 7 to get $a^2 - 7 = 4\sqrt{3}$. Squaring yields that $a^4 - 14a^2 + 49 = 48$, so that $a^4 - 14a^2 + 1 = 0$, and a is a root of $x^4 - 14x^2 + 1$. Again, we can factor this polynomial as

$$x^4 - 14x^2 + 1 = (x^2 - 4x + 1)(x^2 + 4x + 1).$$

3. Solve with radicals by hand (showing your work) the equation

$$x^3 + 6x = 12.$$

One can do this by simply plugging in for p and q in our formulas, but this is not how you do it in practice unless you have the formulas memorized or easily available. For me, I simply remember the picture of the cube that we had, so that I know that I want to set up u and v, where $(u-v)^3 + 3uv(u-v) = u^3 - v^3$. Thinking of $x = u - v$, we want $6 = 3uv$ and $12 = u^3 - v^3$. From here, we solve the first equation for v to get that $v = 2/u$, so that the second equation becomes $12 = u^3 - (2/u)^3$. Multiplying through by u^3, we obtain $12u^3 = u^6 - 8$, so that we want u to satisfy $u^6 - 12u^3 - 8 = 0$. This is a quadratic in u^3, so that the quadratic formula gives us

$$\begin{aligned} u^3 &= \frac{12 \pm \sqrt{(12)^2 - 4(-8)}}{2} \\ &= 6 \pm \frac{1}{2}\sqrt{144 + 32} \end{aligned}$$

$$
\begin{aligned}
&= 6 \pm \frac{1}{2}\sqrt{176} \\
&= 6 \pm \frac{1}{2}\sqrt{16 \cdot 11} \\
&= 6 \pm 2\sqrt{11}
\end{aligned}
$$

Thus, we can set $u = \sqrt[3]{6 + 2\sqrt{11}}$. We can now solve for v^3 by noting that $v^3 = u^3 - 12 = -6 + 2\sqrt{11}$, so that $v = \sqrt[3]{-6 + 2\sqrt{11}}$. Since $x = u - v$, we have

$$x = \sqrt[3]{6 + 2\sqrt{11}} - \sqrt[3]{-6 + 2\sqrt{11}}.$$

Checking this on our calculator suggests that it is correct.

I did the problem this way, to show how one usually solves cubic equations in practice. That is, rather than memorize a complicated formula, it pays to remember the main idea (the picture of the cube), and then solve the problem algorithmically. You can do something very similar with the quadratic formula, where you remember only the picture of the square, and then fill in the numbers as you go along. In particular, you should note how I didn't exactly follow the steps that we did in class. Rather than holding onto the terms $6/2$ and $p/3$, I was able to reduce immediately and use the reduced terms in my formulation.

Compare this to solving a quadratic equation where $a = 1$, and b and c are both even. The quadratic formula requires you to cancel out a 2 along the way, but if you actually solve the equation by completing the square, or using the geometric algorithm with the square, the 2 that needs to be cancelled out of the $\frac{-b}{2}$ term and the square root has been cleaned up.

4. Solve with radicals by hand (showing your work) the equation

$$x^3 + 3x^2 + 15x + 3 = 0.$$

We begin by noting that we wish to eliminate the $3x^2$ term. To do so, we let $x = y - 1$. Then plugging in $y - 1$ for x we obtain

$$y^3 - 3y^2 + 3y - 1 + 3(y^2 - 2y + 1) + 15(y - 1) + 3 = 0,$$

which simplifies to

$$y^3 + 12y - 10 = 0.$$

Since we just did the other process by hand, I will work on this one by the formula. Putting the above equation for y in our "standard form," we obtain $y^3 + 12y = 10$. Thus $p = 12$ and $q = 10$, yielding that $p/3 = 4$, and $q/2 = 5$. Using the formula for the standard cubic that we obtained, we have

$$\begin{aligned} y &= \sqrt[3]{5 + \sqrt{25 + 64}} - \sqrt[3]{-5 + \sqrt{25 + 64}} \\ &= \sqrt[3]{5 + \sqrt{89}} - \sqrt[3]{-5 + \sqrt{89}}. \end{aligned}$$

As $x = y - 1$, we have that

$$x = -1 + \sqrt[3]{5 + \sqrt{89}} - \sqrt[3]{-5 + \sqrt{89}}.$$

5. It is unknown whether $e\pi$ and $e + \pi$ are transcendental or not. Curiously, it is known that at least one of them must be transcendental. Use the following outline to prove this:

   (a) Show that either $e+\pi$ or $e-\pi$ is transcendental. (You are allowed to use that e and π are transcendental, and that the algebraic numbers form a field.)

   (b) Calculate $(e + \pi)^2 - 4e\pi$ and factor.

   (c) Using that the algebraic numbers are closed under square roots, show that if both $e+\pi$ and $e\pi$ are algebraic then $e - \pi$ would also be algebraic.

   (d) Using the first part, show that either $e+\pi$ or $e\pi$ is transcendental.

I will write this proof in one short proof, rather than do the step-by-step outline.

Let us begin by assuming that $e+\pi$ is algebraic (otherwise there would be nothing to prove). If $e-\pi$ were also algebraic, then $e+\pi+(e-\pi) =$

2π would also be algebraic since the algebraic numbers are closed under addition. As the algebraic numbers are also closed under multiplication, this would imply that $\frac{1}{2} \cdot 2\pi = \pi$ was algebraic, which is a contradiction. Consequently, $e - \pi$ must be transcendental in this case.

Since $(e + pi)^2 - 4e\pi = e^2 - 2e\pi + \pi^2 = (e - \pi)^2$, if both $e + pi$ and $4e\pi$ were algebraic, then $(e - \pi)^2$ would also have to be algebraic. However, the algebraic numbers are closed under square roots since if α is a root of $p(x)$, then $\sqrt{\alpha)}$ is a root of $p(x^2)$ which is a polynomial if $p(x)$ is. Consequently, if both $e + \pi$ and $e\pi$ are algebraic, then $e - \pi$ would have to be algebraic. This, however, is a contradiction to the first paragraph, so if $e + \pi$ is algebraic, then it must be the case that $e\pi$ is not algebraic and is thus transcendental.

6. Discuss the difference between transcendental and algebraic numbers and how this affects a student's ability to understand each. How might this influence your teaching about π and e?

   An algebraic number is a number that is the root of a polynomial with integer coefficient. A real number is transcendental if it is not algebraic. Thus, a transcendental number is not the root of any polynomial with integer coefficients. In addition to making these concepts opposites, this also means that a transcendental number is defined negatively, in that it is only be defined by properties that it doesn't satisfy rather than by properties that it does satisfy. When students encounter algebraic numbers, they have a pretty solid foundation for working with it. Graphing the polynomial by hand or on a calculator, they can zero in on where the roots of the polynomial lie, and pretty quickly, they can assign the number to a place on the number line. They also have one of the main properties that defines this number at their fingertips, namely that it is a root of the polynomial. For example, $\sqrt{-1}$ despite being a complex number, is relatively easy for students to deal with algebraically, even if they have some concerns over what it represents. In contrast to algebraic numbers, transcendental numbers must be defined by properties that the students are less likely to be familiar with. For example, transcendental number that students might see (without ever hearing that it is transcendental) is $2^{\sqrt{2}}$. This number is much harder to zero in on. Certainly, the student's calculator can calculate an approximation for it, but for the most part, the **decimal approximation** of $2^{\sqrt{2}}$ is how the students will define the number internally, as opposed to the number $\sqrt{2}$, which the students will internally define as a number that squares to 2.

Thus, when introducing students to transcendental numbers, the teacher has two basic alternatives. The first is to introduce the number and try and have the students think about it by its decimal approximation, and the second is to give the students some other defining property. As I believe that students are far too likely to treat decimal expressions as the definition of a number (leading to difficulties with $.\overline{9} = 1$), it seems to me that the teacher should choose the second alternative whenever possible.

For π this isn't too bad. Most students will be introduced to π early on as the special number that helps us find the area of a circle. They also will hear of it as being defined as the ratio of the circumference to the diameter of a circle. Consequently, when teaching π, you can emphasize the property, and then come back to the idea of approximations.

The number e on the other hand is a problem this way. While many classical texts introduce e via instantaneously compounding interest, the very next statement is that e is the number that is approximately $2.71828182845945\ldots$. This, of course, leads the students to treat e as its decimal approximation. The main problem with this is that e becomes mostly unmotivated to the students, and thus it becomes a number that they treat as one of those silly things that math teachers force us to learn. So, what properties could we use to introduce e? We want them to somehow be natural properties for students to think about. One high school math textbook uses two such properties, that e is the number so that the function e^x has slope 1 at $x = 0$, and that e is the number so that the area under the graph of $\frac{1}{x}$ between 1 and e is equal to 1. From the advanced viewpoint, we recognize that both of these definitions require some fundamental understanding of calculus. However, neither actually requires calculus. The first requires that the students understand the idea of slopes of tangent lines, which is really the idea of rate of change. The second requires that the students understand the idea of area. The trick is that when we divide algebra and geometry, defining a number by area makes some students feel uneasy. The number π is typically introduced when students are used to thinking about many areas of mathematics at once, so it doesn't run into this problem, while the number e does.