Solutions
Homework Set #9
The Transcendentality Proof

To begin with, trying to teach high school students about why any number is transcendental is at some level an impossible task. Thus, before getting started, you really need to decide what would be the purpose of such a task. For me, the purpose is two-fold: first, I would like the students to gain some little understanding of the history of the discovery of transcendental numbers and of the difficulties of proving statements like e and π are transcendental; second, a basic discussion of this proof can provide the students with a summary of many of the important topics they have learned, and how they might fit together. Thus, my presentation will be aimed at achieving these purposes, as I see them.

The topics I will spend some time on are: rate of change, absolute value and distance, and working with rational numbers.

We have briefly mentioned before that e and π are not only irrational numbers, but that they are even more complicated. That is, they are transcendental. Since no one recalls what a transcendental number is, let me refresh everyone's memory about the various types of numbers. When we look at real numbers, there are two main classes that we look at, the rational numbers and the irrational numbers. The irrational numbers, the numbers that cannot be written as fractions of integers, also break into two pieces, the algebraic numbers and the transcendental numbers. The algebraic numbers consist of both rational numbers, and those irrational numbers that are the roots of "nice" polynomials, namely those with integer coefficients. For example, $\sqrt{2}$ is algebraic since it is a root of the polynomial $x^2 - 2$. Another way of looking at this property, is that for an algebraic number you can add and subtract together various of its powers to get a non-zero integer. For example, if we take the number $a = \sqrt{2} + \sqrt{3}$, we have that $a^2 = 5 + 2\sqrt{6}$, $a^3 = 11\sqrt{2} + 9\sqrt{3}$, and $a^4 = 49 + 20\sqrt{6}$. Thus

$$a^4 - a^2 - a^2 - a^2 - a^2 - a^2 - a^2 - a^2 - a^2 - a^2 - a^2 = a^4 - 10a^2 = -1.$$

We say that a number is transcendental if it is not algebraic. That is, a number is transcendental if it is not the root of any polynomial with integer coefficients, or perhaps more directly, if we cannot add together powers of the number to get an integer out, **no matter how we try to do it**. One of the problems with this definition, is that it is negative. That is, to show that something is transcendental requires eliminating an infinite number of possibilities. Thus, proving any number is transcendental is apt to be very hard. Historically, Euler defined the concept of transcendental numbers in the 18th century. However, he **couldn't show that any number was transcendental!** It took 60 years before anyone could prove that transcendental numbers existed. It took another 40 years to prove that π and e are transcendental, and the proof for π required using the complex numbers!

Today, I am going to discuss the proof that the number

$$.11000100000000000000000100000000000000 \ldots 010 \ldots 01 \ldots$$

is transcendental. This number has 1s in the $n!$ digits of its decimal expansion, and 0s everywhere else. Mathematicians write this number as

$$\sum_{n=1}^{\infty} a10^{-n!}.$$

This number was the first number to be shown to be transcendental, and that was done by Liouiville in the 1840s. We will not go into the full details of the proof, but rather we shall outline the main ideas of the proof and relate them to the ideas we have been studying in class.

First, let us consider the graph of any polynomial. We know from our experience that the graph is never vertical. Another way of saying this is that for any finite portion of the graph, and any point on the graph, we can sandwich the graph between two lines to get the following type of picture.

This is simply saying that the rate of change of any polynomial is limited.

At this point, I need to tell you what we are going to do. We are planning to show that every algebraic number has the special property that you can only get so close to it with rational numbers. Then we shall show that this funny number above violates this property. Thankfully, the property that we are talking about is negative for algebraic numbers (something doesn't happen infinitely often), so that it will be positive for our number. The particular property is that for any algebraic number α that is the root of a polynomial of degree n, there are only finitely many rational numbers $\frac{a}{b}$ (where a and b are integers) such that

$$|\frac{a}{b} - \alpha| < \frac{1}{b^{n+1}}. \tag{1}$$

Before going on to explain how you might establish this property, let's look at an example. If $\alpha = \sqrt{2}$, this says that there are only finitely many fractions with

$$|\frac{a}{b} - \sqrt{2}| < \frac{1}{b^3}.$$

For example, $\frac{1}{1}$ and $\frac{2}{1}$ are two such fractions since $\frac{1}{1^3} = 1$. However, if we had a denominator of 2, we would have to be within $\frac{1}{8}$th of $\sqrt{2}$, and the only possibility is $\frac{3}{2} = 1.5$ which is within $\frac{1}{10}$ of $\sqrt{2}$. However, if the denominator is 3, then we need to be within $\frac{1}{27} = .\overline{037}$, and there are no choices for the numerator. One example doesn't close the book on the possibility, but I think we have enough of an idea of what is going on.

In the proof, what we shall do is use the slopes of the lines that sandwich the graph of the polynomials $p(x)$ to show that if

$$|\frac{a}{b} - \alpha| < \frac{1}{|m|b^n},$$

where $|m|$ is the larger of the slopes of the bounding lines (in absolute value), then $|p(a/b)| < \frac{1}{b^n}$. From this, we shall establish that $p(a/b) = 0$ using least common denominators. If $b > m$, however, this tells us that there are only finitely many denominators that might work, and from there, it is easy to believe (but not quite as easy to show) that there are only finitely many fractions that can satisfy equation **??**.

From here on, we shall do a special case, where $p(x)$ has degree 3 (so that $n = 3$). To show the above, consider that the graph of $p(x)$ is sandwiched between two lines, one of slope m and one of slope $-m$ (where we take $m > 0$). Then $p(a/b)$ is between the y-values that these lines take on at a/b. However, since $p(\alpha) = 0$, this tells us that

$$-m < \frac{p(a/b)}{|\frac{a}{b} - \alpha|} < m$$

using the rise over run calculation of the slope of a line.

Thus clearing the denominator, we have

$$\frac{-m}{|\frac{a}{b}-\alpha|} < p(a/b) < \frac{m}{|\frac{a}{b}-\alpha|}.$$

Consequently, $|p(a/b)| < |\frac{m}{\frac{a}{b}-\alpha}|$. If we now suppose that

$$|\frac{a}{b}-\alpha| < \frac{1}{mb^3},$$

it follows that $|p(a/b)| < \frac{1}{b^3}$, as I told you we would show. (Now, there was nothing fancy about our 3 in the exponent of b. We only used it at the end, so we could just as well make $p(x)$ of degree n and then we would get what we were looking for before.)

For the next part, let's assume that $p(x) = 2 - 4x + 5x^2 - 6x^3$. Then we would have

$$\begin{aligned} p(a/b) &= 2 - 4a/b + 5(a/b)^2 - 6(a/b)^3 \\ &= 2 - \frac{4a}{b} + \frac{5a^2}{b^2} - \frac{6a^3}{b^3} \\ &= \frac{2b^3}{b^3} - \frac{4ab^2}{b^3} + \frac{5a^2b}{b^3} - \frac{6a^3}{b^3} \\ &= \frac{2b^3 - 4ab^2 + 5a^2b - 6a^3}{b^3}. \end{aligned}$$

This last is a fraction, and since a and b are necessarily integers, either the numerator is 0 or it is greater than or equal to 1 in absolute value. Thus, if $|p(a/b)| < \frac{1}{b^3}$, it must be the case that the numerator is 0. (I would try to spend a lot of time here, making sure the students understood this point.) Of course, if $p(x)$ has degree n and the coefficients are any integers, we would simply get that if $|p(a/b)| < \frac{1}{b^n}$, then $p(a/b) = 0$. However, $p(x)$ can have only finitely many roots, so there can only be finitely many rational numbers $\frac{a}{b}$ such that $b > m$ and $|\frac{a}{b} - \alpha| < \frac{1}{b^{n+1}}$. Note that for $\frac{a}{b}$ to satisfy this inequality implies that it lies within an interval of length $\frac{2}{b^{n+1}}$ on the number line

so that at most one or two numbers with any particular denominator b can satisfy the inequality when $b < m$. Thus, all together, only finitely many rational numbers can satisfy this inequality for α a root of $p(x)$.

At this point, we simply want to show that no matter how you choose n and m, you can find infinitely many rational numbers satisfying this inequality for $\alpha = .11000100\ldots$. Let us do the case where $m = 5$ and $n = 3$. If we look at the number $y = .110001$, then $y = \frac{110001}{1000000} = \frac{110001}{10^6}$, and

$$\alpha - y = .00000000000000000000000100\ldots01\ldots < \frac{2}{10^{24}}.$$

Since $(10^6)^4 = 10^{24}$, our denominator is not quite small enough, but if instead we took for y, the first 24 digits of α, we would similarly have that

$$\alpha - y < \frac{2}{10^{120}}.$$

As y can be written as a fraction with denominator 10^{24}, and $(10^{24})^4 = 10^{96} < 10^{120}$, we would have that y is "too close" to α, so that $p(y) = 0$. However, if we pick y to be all the digits up to any 1 beyond the 4th one, we will run into a similar calculation, and thus all of these infinitely many choices for y give a rational number satisfying equation **??**, and our number α cannot be algebraic.

I don't really expect that many of you followed all of this, but I want you to look at it for a moment in a piece-by-piece fashion. There isn't really anything that we haven't talked about. The careful proof requires calculus, but that is only in the case where you look at the lines bounding the polynomial. Amazingly, the idea of how close we can get to a number with fractions is a major part of mathematical number theory, and from it, we get approximations for numbers like π. What is even more amazing and annoying, is that to show that π or e is transcendental is significantly harder than showing that α is transcendental. This arises from the fact that so far as we know, you cannot approximate e or π this well.