

## Solutions and Comments on Homework 6

1. Show that  $\sqrt[3]{25}$  cannot be written as  $a + b\sqrt[3]{5}$  for any rational numbers  $a$  and  $b$ .

**Proof 1:** Suppose otherwise, and let  $\sqrt[3]{25} = a + b\sqrt[3]{5}$  for some rational numbers  $a$  and  $b$ . Then multiplying both sides of this equation by  $\sqrt[3]{5}$ , we obtain  $5 = a\sqrt[3]{5} + b\sqrt[3]{25}$ . Substituting our first equation into this equation (for  $\sqrt[3]{25}$ ), we obtain

$$5 = a\sqrt[3]{5} + b(a + b\sqrt[3]{5}).$$

Thus we have the equation  $5 - ab = (a + b^2)\sqrt[3]{5}$ . As we are assuming that we know that  $\sqrt[3]{5}$  is irrational, if  $a + b^2 \neq 0$ , we would have a contradiction, since  $\sqrt[3]{5}$  would equal  $\frac{5-ab}{a+b^2}$ . Consequently,  $5 - ab = 0$  and  $a + b^2 = 0$ . The second equation tells us that  $a = -b^2$ . The first equation then implies that  $5 + b^3 = 0$ , so that  $b = -\sqrt[3]{5}$ . However, as  $b$  is a rational number, this is a contradiction. Thus our original assumption must have been false, and the result follows.

**Proof 2:** We begin by noting that  $x^3 - 5$  is an irreducible polynomial over  $Q$ , since it has no rational roots and is of only degree 3. If  $\sqrt[3]{25} = a + b\sqrt[3]{5}$ , however, then  $\sqrt[3]{5}$  is a root of the polynomial  $x^2 - bx - a$ . Thus, it follows by the root theorem that  $x - \sqrt[3]{5}$  is a factor of this polynomial (over the real numbers). Thus, it is a factor of the greatest common divisor of  $x^3 - 5$  and  $x^2 - bx - a$  (over  $Q$ ). But since  $x^3 - 5$  is irreducible and  $x^2 - bx - a$  is of degree smaller than 3, their greatest common divisor (over  $Q$ ) must be 1. Clearly, though,  $\sqrt[3]{5}$  is not a root of the constant polynomial 1, so we have reached a contradiction. Thus, our assumption must not be true and the result follows.

2. Find the multiplicative inverse of the number  $x = 1 + 2\sqrt[3]{3} + 2\sqrt[3]{9}$ , and prove that your answer is correct.
  - (a) Using the basis  $\{1, \sqrt[3]{3}, \sqrt[3]{9}\}$  for the field  $Q[\sqrt[3]{3}]$  over the field  $Q$ , write the  $3 \times 3$  matrix  $M$  corresponding  $x$  viewed as a linear transformation from  $Q^3$  to  $Q^3$ .

- (b) Use Gaussian elimination (or your calculator) to find the inverse of the matrix  $M$ .
- (c) Find the element  $y$  of  $Q[\sqrt[3]{3}]$  corresponding to  $M$ . Now multiply  $x$  and  $y$  to check that  $xy = 1$ .

Using the basis  $\{1, \sqrt[3]{3}, \sqrt[3]{9}\}$ , we see that an arbitrary element  $a + b\sqrt[3]{3} + c\sqrt[3]{9}$  corresponds to the matrix

$$\begin{pmatrix} a & 3c & 3b \\ b & a & 3c \\ c & b & a \end{pmatrix}.$$

Using our graphing calculator and the values  $a = 1$ ,  $b = 2$ , and  $c = 2$ , we obtain the inverse matrix. Unfortunately, our calculator gives us decimals rather than fractional entries. To solve this problem, we multiply the calculator entries by the determinant of our original matrix to get that the inverse matrix is

$$\frac{1}{61} \begin{pmatrix} -11 & 6 & 30 \\ 10 & -11 & 6 \\ 2 & 10 & -11 \end{pmatrix}.$$

Thus the inverse of the given element would be

$$\frac{-11}{61} + \frac{10}{61}\sqrt[3]{3} + \frac{2}{61}\sqrt[3]{9}.$$

We can now carry out the multiplication (by hand) to check that this is correct, and I encourage you to do so.

You can actually use other matrix rules to find the general inverse of the matrix. A rule that used to be commonly taught for finding inverses states that the inverse of a matrix can be found by taking determinants of the minors. Using this trick, we get that the generic inverse of an element  $a + b\sqrt[3]{3} + c\sqrt[3]{9}$  should be

$$\frac{1}{a^3 + 3b^3 + 9c^3 - 9abc} (a^2 - 3bc + (3c^2 - ab)\sqrt[3]{3} + (b^2 - ac)\sqrt[3]{9}).$$

To check out this solution, we multiply it by  $a + b\sqrt[3]{3} + c\sqrt[3]{9}$ , and we get out

$$\frac{1}{a^3 + 3b^3 + 9c^3 - 9abc} \begin{pmatrix} a^3 + 3c^3 - 3abc + 3b^3 - 3abc + \\ (a^2b - 3b^2c + 3ac^2 - a^2b + 3b^2c - 3ac^2)\sqrt[3]{3} \\ +(a^2c - 3bc^2 + 3bc^2 - ab^2 + ab^2 - a^2c)\sqrt[3]{9} \end{pmatrix}.$$

Simplifying this we do indeed get 1.

3. If we take the basis of  $\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}$  for the field  $Q[\sqrt{3}, \sqrt{5}]$  over  $Q$ , then what is the matrix associated to the element  $2 + \sqrt{3} - \sqrt{15}$  of this ring? What is the inverse of this element?

Working as before, we have that the matrix corresponding to an arbitrary element here is

$$M = \begin{pmatrix} a & 3b & 5c & 15d \\ b & a & 5d & 5c \\ c & 3d & a & 3b \\ d & c & b & a \end{pmatrix}.$$

Using  $a = 2$ ,  $b = -1$ ,  $c = 0$ , and  $d = -1$ , and asking the calculator to give the inverse, we get

$$M^{-1} = \frac{-1}{4} \begin{pmatrix} 7 & -12 & -15 & 30 \\ -4 & 7 & 10 & -15 \\ -3 & 6 & 7 & -12 \\ 2 & -3 & -4 & 7 \end{pmatrix}.$$

Thus the inverse element is  $\frac{-7}{4} + \sqrt[3]{+\frac{3}{4}}\sqrt{5} - \frac{1}{2}\sqrt{15}$ , as can now be checked by multiplying out by hand.

We will use the next several problems in our proof of the impossibility of constructing  $\sqrt[3]{2}$  and of trisecting a sixty degree angle with straightedge and compass. First, though, we need some definitions:

Assume that  $P_0$  is a set of points in the Euclidean plane  $R^2$ , and consider operations of two kinds:

- (a) through any two points draw a straight line, and
- (b) Draw a circle centered at one point and through any other point.

A point  $(x, y) \in R^2$  is said to be *constructible* from  $P_0$ , if there exist a sequence of points

$$(x_1, y_1), (x_2, y_2), \dots, (x_t, y_t) = (x, y)$$

such that the point  $(x_i, y_i)$  can be obtained as the intersection point of two distinct lines or a line and a circle or two distinct circles when drawn using one of our two operations from the points  $P_0 \cup \{(x_1, y_1), \dots, (x_{i-1}, y_{i-1})\}$ .

We say that  $(x, y)$  is *constructible over the rationals* if  $(x, y)$  is constructible over the set  $P = \{(a, b) \mid a, b \in Q\}$ .

Our first goal is to show that a point  $(x, y)$  is constructible over  $Q$  if and only if the numbers  $x$  and  $y$  are constructible by our previous definition.

4. In one of the previous homework sets, we showed the constructible numbers form a field. Show that this implies that for any positive rational number  $x$ , a segment of length  $x$  can be constructed given a segment of unit length. Note: you do not have to describe how to construct  $x$ , simply describe the intermediate steps. (For example,  $\frac{2}{3}$  is constructible since using addition, we know that segments of length  $1 + 1 = 2$  and  $1 + 1 + 1 = 3$  are constructible. Using division, we know that a segment of length  $\frac{2}{3}$  is constructible.)

We modify the above procedure. If  $x$  is a positive rational number, then  $x = \frac{a}{b}$ , where  $a$  and  $b$  are two positive integers. As 1 is constructible and the constructible numbers form a field, closure under addition implies that  $a = (1 + \dots + 1)$  and  $b = (1 + \dots + 1)$  are both constructible. Now, the closure under division (which can be applied as  $b \neq 0$ ) implies that  $\frac{a}{b}$  can be constructed, and thus we have constructed  $x$ .

5. Show that given segments of length  $x$  and  $y$ , then you can construct the point  $(x, y)$  in the Cartesian plane.

Ideally, this should include the constructions, but I don't have the time right now to input them, so I will simply list the steps and the justification.

We are given the origin  $(0, 0)$  and the  $x$ - and  $y$ -axes, together with line segments of length  $x$  and  $y$ . Transport the length  $x$  to find the point  $(x, 0)$  on the  $x$ -axis by taking a circle of radius  $x$  centered at the origin. Similarly, transport the distance  $y$  to the  $y$ -axis to find the point  $(0, y)$ . At this point we can take two perpendiculars to these points (using the method given in class) or we can use an idea of Emily's where we draw a circle centered at  $(0, y)$  of radius  $x$  and a circle centered at  $(x, 0)$  of radius  $y$ , and look for their intersection points, one of which will be the desired point  $(x, y)$  (which must occur in the first quadrant). This technique requires some proof, but the basic idea is that we have constructed a rectangle, and we just need to prove that it really is a rectangle.

6. Show that given a point  $(x, y)$  in the Cartesian plane, then you can construct segments of length  $x$  and  $y$ . (For this problem, you will need to show how you construct a perpendicular from a given point to a given line (in your case, from  $(x, y)$  to the  $x$ -axis or the  $y$ -axis).

For this, we need to construct a line through  $(x, y)$  that is perpendicular to the  $x$ -axis. For this, construct a circle centered at  $(x, y)$  intersecting the  $x$ -axis at two points labeled  $A$  and  $B$ . Now construct the circles  $C_A$  and  $C_B$  centered at  $A$  and  $B$  respectively that go through  $(x, y)$ . From the classroom construction, the line  $\ell$  connecting the two points of intersection of  $C_A$  and  $C_B$  is a perpendicular bisector of the segment  $\overline{AB}$ , a segment of the  $x$ -axis. Thus the coordinates of the intersection point  $D$  of  $\ell$  and the  $x$ -axis are  $(x, 0)$ , and we have constructed a segment of length  $x$ . We can similarly construct a segment of length  $y$ . (Alternatively, the segment from  $D$  to  $(x, y)$  has length  $y$ .)

The last two problems showed that the point  $(x, y)$  is constructible over  $Q$  if and only if you can construct segments of length  $x$  and  $y$  given a segment of length 1. Thus they showed that we can define a constructible number as a coordinate of a constructible point over  $Q$ .

We now want to show that any time you construct a new point, you can get its coordinates from the coordinates of the points defining the given circle(s) and/or line(s) by using addition, subtraction, multiplication, division, and square roots.

7. Let  $l_1$  be a line through the points  $(a, b)$  and  $(c, d)$  and  $l_2$  be a line through the points  $(x, y)$  and  $(z, w)$ . Find the coordinates of the intersection point of these two lines (assuming such a point exists).

The main difficulty that students had with this problem was not covering the case where the slope of one of the lines was equal to 0. This happens because we are often wedded to the point-slope formula of a line as the most useful. There are times, however, when other versions are useful, and in particular in this case, the standard form of a line, namely  $Ax + By = C$ , as all lines can be represented this way for some choice of  $A$ ,  $B$ , and  $C$ . We will begin by solving for the intersection points of two lines  $Ax + By = E$  and  $Cx + Dy = F$ , and then we will figure out what the values of our coefficients are in this case.

To solve this, we could use linear algebra and  $2 \times 2$  matrices to find the solutions to these two linear equations. Doing it this way, we obtain that the solutions are

$$\begin{aligned}x &= \frac{1}{AD - BC}(DE - BF) \\y &= \frac{1}{AD - BC}(AF - CE)\end{aligned}$$

which you can check out by plugging them in. (Alternatively, you can multiply the top equation by  $C$  and the bottom equation by  $-A$  and then add the two equations to obtain a single equation for  $Y$  and then solve this.) Now, our solution has one possible flaw, which is that  $AD - BC$  might equal 0. In this case, however, the slopes of the lines are  $\frac{-B}{A}$  and  $\frac{-D}{C}$ , and these are equivalent fractions if and only if  $AD - BC = 0$ . That is,  $AD - BC = 0$  if and only if the slopes of the two lines are the same, and then there is no unique point of intersection as they are either parallel or the same line. Given these solutions, we now solve for  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $F$ . We have the equations

$$\begin{aligned}Aa + Bb &= E && \text{and} \\Ac + Bd &= E.\end{aligned}$$

I know that  $A(x - a) + B(y - b)$  should equal 0 for all  $x$  and  $y$  on the line, which allows for us to rid ourselves of  $E$  for the time being. If  $(c, d)$  is plugged in here, we then have that  $A(c - a) = B(b - d)$ , and

at least one of  $c - a$  and  $b - d$  is non-zero. A clear solution to this last is  $A = (b - d)$  and  $B = (c - a)$ . As any solution must be a multiple of this one, we are free to choose this one (as the multiple would factor out of the final equation). Thus,  $E$  should satisfy

$$(b - d)(a) + (c - a)(b) = E,$$

or  $E = bc - ad$ , and the equation of our line is

$$(b - d)x + (c - a)y = bc - ad.$$

I could have just said this and checked that it went through our points, but then you wouldn't know how to get it.

Thus, our choices are  $A = (b - d)$ ,  $B = (c - a)$ , and  $E = (bc - ad)$ . Working similarly, we get  $C = (y - w)$ ,  $D = (z - x)$ , and  $F = yz - xw$ . Plugging these in to

$$\begin{aligned} x &= \frac{1}{AD - BC}(DE - BF) \\ y &= \frac{1}{AD - BC}(AF - CE) \end{aligned}$$

we get

$$\begin{aligned} x &= \frac{1}{(b - d)(z - x) - (c - a)(y - w)}((z - x)(bc - ad) - (c - a)(yz - xw)) \quad \text{and} \\ y &= \frac{1}{(b - d)(z - x) - (c - a)(y - w)}((b - d)(yz - xw) - (y - w)(bc - ad)). \end{aligned}$$

8. Write the equation of a circle centered at the point  $(a, b)$  going through the point  $(c, d)$ .

$$(x - a)^2 + (y - b)^2 = (c - a)^2 + (d - b)^2.$$

At this point, it becomes worthwhile to simplify the upcoming notational nightmare by using translations of the plane to make sure that if we have circles to intersect, then at least one of them is centered at the origin. To do this, we need to show that we can translate entire problems without causing trouble to our definitions of constructible numbers and points.

9. Show that if  $(c, d)$  is a point lying at the intersection of a circle centered at  $(a, b)$  through  $(e, f)$  and a line through  $(g, h)$  and  $(j, l)$ , then  $(c-a, d-b)$  lies at the intersection of the circle centered at  $(0, 0)$ ,  $(e-a, f-b)$  and the line through  $(g-a, h-b)$  and  $(j-a, l-b)$ . Make a similar statement for the case where we have the intersection of two circles.

We begin by writing the equations of the first given line and circle. The circle centered at  $(a, b)$  through  $(e, f)$  has equation

$$(x - a)^2 + (y - b)^2 = (e - a)^2 + (f - b)^2.$$

The line through  $(g, h)$  and  $(j, l)$  has equation

$$y - h = \frac{l - h}{j - g}(x - g),$$

assuming  $j - g \neq 0$ . Then  $(c, d)$  is an intersection point if and only if  $(c, d)$  satisfies both equations. That is

$$(c - a)^2 + (d - b)^2 = (e - a)^2 + (f - b)^2 \quad \text{and} \quad (1)$$

$$c - h = \frac{l - h}{j - g}(d - g). \quad (2)$$

Thus, we may assume that both of these equations are true.

The circle centered at the origin through  $(e-a, f-b)$  then has equation

$$x^2 + y^2 = (e - a)^2 + (f - b)^2,$$

and the line through  $(g-a, h-b)$  and  $(j-a, l-b)$  has equation

$$y - (h - b) = \frac{l - b - (h - b)}{j - a - (g - a)}(x - (g - a)).$$

Thus, to see if  $(c-a, d-b)$  is an intersection point of this circle and line, it suffices to see if this point satisfies both equations. Plugging the value into the first equation, we get

$$(c - a)^2 + (d - b)^2 = (e - a)^2 + (f - b)^2,$$

which is true from the above discussion showing  $(c, d)$  is on the original circle. Plugging the value into the equation of our new line, we get

$$(d - b) - (h - b) = d - b$$

on the left hand side and

$$\frac{l-h}{j-g}(c-a-(g-a)) = \frac{l-h}{j-g}(c-a)$$

on the right hand side. We know that these two quantities are equal, however, by equation ?? above.

The case where  $j-g=0$  works similarly, except that our lines have much simpler equations, namely  $x=g$  and  $x=g-a$ .

The statement for the case of two circles is that if  $C_1$  is the circle centered at  $(a,b)$  through  $(c,d)$  and  $C_2$  is a circle centered at  $(e,f)$  through  $(g,h)$ , then  $(k,l)$  is an intersection point of  $C_1$  and  $C_2$  if and only if  $(k-a,l-b)$  is an intersection point of the circle centered at the origin through the point  $(c-a,d-b)$  and the circle centered at  $(e-a,f-b)$  through  $(g-a,h-b)$ .

10. Given the above, argue that to see that we only need to use the operations of addition, subtraction, multiplication, division, and taking square roots to find intersection points of circles and lines, it suffices to show the condition where one of the circles is centered at the origin.

If we knew that we had this condition where one circle is centered at the origin, since we can recover the intersection point of arbitrary circles by simply adding the coordinates of the appropriate new center point, and these coordinates are in the field, then we have not added any new operations to the coordinates of the points of intersections.

11. Let  $C$  be a circle centered at the origin through the point  $(a,b)$ , and let  $l$  be a line through the points  $(c,d)$  and  $(e,f)$ . Find the coordinates of the points of intersection of  $C$  and  $l$  if they exist.

In this case,  $C$  has equation  $x^2 + y^2 = r^2$ , where  $r^2 = a^2 + b^2$ . First, let us suppose that  $c=e$ , so that the line has equation  $x=c$ . In this case, the intersection points of the line and the circle are given by  $(c, \pm\sqrt{r^2 - c^2}) = (c, \pm\sqrt{a^2 + b^2 - c^2})$  if they exist. (If the quantity under the square root sign is negative, then no solutions exist.)

Otherwise, we may assume that the equation of the line is given by

$$y-d = \frac{f-d}{e-c}(x-c).$$

In this case, letting  $m = \frac{f-d}{e-c}$  be the slope of the line, we have that  $y = mx + d - mc$ . Writing  $k = d - mc$  be the  $y$ -intercept, we then can substitute into our equation for the circle and we obtain the equation

$$x^2 + (mx + k)^2 = r^2.$$

Expanding this we get

$$(1 + m^2)x^2 + 2mkx + (k^2 - r^2) = 0.$$

Using the quadratic formula, we obtain that

$$x = \frac{-2mk \pm \sqrt{4m^2k^2 - 4(1 + m^2)(k^2 - r^2)}}{2(1 + m^2)}.$$

Simplifying this, we obtain

$$\begin{aligned} x &= \frac{-mk \pm \sqrt{(1 + m^2)r^2 - k^2}}{1 + m^2} \quad \text{and} \\ y &= m \frac{-mk \pm \sqrt{(1 + m^2)r^2 - k^2}}{1 + m^2} + k. \end{aligned}$$

The interested reader is welcome to plug all of the variables back in. You should note, however, that  $m$ ,  $k$ , and  $r^2$  are all elements of the same field as the coordinates of the original points are in, so that all we used to get our new coordinates were the old coordinates and the operations of addition, subtraction, multiplication, division, and the taking of exactly one new square root.

12. Let  $C_1$  be a circle centered at the origin through the point  $(a, b)$ , and let  $C_2$  be a circle whose center is at the point  $(c, d)$  and goes through the point  $(e, f)$ . Find the intersection points of  $C_1$  and  $C_2$  if they exist. The equations of our two circles are

$$\begin{aligned} x^2 + y^2 &= r_1^2 \quad \text{and} \\ (x - c)^2 + (y - d)^2 &= r_2^2, \end{aligned}$$

where  $r_1^2 = a^2 + b^2$  and  $r_2^2 = (e - c)^2 + (f - d)^2$ . Subtracting the second equation from the first, we get that the intersection points of these two circles must lie on the line

$$2cx - c^2 + 2dy - d^2 = r_1^2 - r_2^2.$$

Assuming  $d \neq 0$ , this line can be written as

$$y = \frac{c}{d}x + c^2 + d^2 + r_1^2 - r_2^2.$$

Writing  $m = \frac{c}{d}$  and  $k = c^2 + d^2 + r_1^2 - r_2^2$ , we now have that  $y = mx + k$  where  $m$  and  $k$  are elements of the field the coordinates of our original points are in.

Now, referring back to the previous problem, the intersection points of this line and the circle  $x^2 + y^2 = r_1^2$  is given by

$$\begin{aligned} x &= \frac{-mk \pm \sqrt{(1+m^2)r_1^2 - k^2}}{1+m^2} \quad \text{and} \\ y &= m \frac{-mk \pm \sqrt{(1+m^2)r_1^2 - k^2}}{1+m^2} + k. \end{aligned}$$

Again, these new coordinates only involved the old coordinates, the four operations and the taking of square roots. Note, if  $d = 0$ , then we are in the case where the line has the form  $x = K$  for some constant  $K$ .

You have now shown that to find new intersection points only requires using the old field operations and square roots.

The next thing is to analyze trigonometry to see how this will affect the trisection.

13. State the angle sum formula for the cosine function.

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

14. Use this to express  $\cos(3\alpha)$  in terms of  $\cos(\alpha)$  and  $\sin(\alpha)$ , and then use the Pythagorean relation  $\sin^2(\alpha) = 1 - \cos^2(\alpha)$  to express  $\cos(3\alpha)$  entirely in terms of  $\cos(\alpha)$ .

Using that  $3\alpha = \alpha + 2\alpha$ , we have  $\cos(3\alpha) = \cos(\alpha)\cos(2\alpha) - \sin(\alpha)\sin(2\alpha)$ . Using the double angle formulas,  $\cos(2\alpha) = 2\cos^2(\alpha) - 1$  and  $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ , we obtain that

$$\cos(3\alpha) = 2\cos^3(\alpha) - \cos(\alpha) - 2\sin^2(\alpha)\cos(\alpha).$$

At this point we use the Pythagorean identity  $\sin^2(\alpha) = 1 - \cos^2(\alpha)$ , and we get

$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha).$$

This is the desired result.

15. What is  $\cos(60)$  (the angle is measured in degrees)? Show that the  $\cos(20)$  is the root of a cubic polynomial with integer coefficients using the previous problem.

As  $\cos(60) = \frac{1}{2}$ , we note that  $\cos(20)$  must solve the polynomial  $f(x) = 8x^3 - 6x - 1$  by the previous problem, since  $8\cos^3(20) - 6\cos(20) = 2\cos(60) = 1$ .

16. Show that the polynomial above has no rational roots.

By the rational root theorem, possible rational roots are  $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$ , and  $\pm \frac{1}{8}$ . A simple check now yields that

$$\begin{aligned} f(-1) &= -3 \\ f(1) &= 1 \\ f(-1/2) &= 1 \\ f(1/2) &= -3 \\ f(-1/4) &= -1/8 + 3/2 - 1 = 3/8 \\ f(1/4) &= -19/8 \\ f(-1/8) &< 0 \quad \text{and} \\ f(1/8) &< 0. \end{aligned}$$

Hence, as none of these are 0,  $f(x)$  can have no rational roots.