

## Solutions and Comments Homework #5

1. Look up in an abstract algebra book the definitions of an integral domain. Give a step by step solution to the problem. Find all  $x$  such that

$$x^2 + 3x + 2 = 0.$$

For each step, state the property of rings or integral domains that allows you to make that step. Comment: The problem was meant to say over the integers, but this was inadvertently left out. Of course, the solution works for any integral domain  $D$ , except that you need to have the proviso that  $n$  really represents  $1_D + \dots + 1_D$  where  $1_D$  is the multiplicative identity for  $D$ .

We will do this in two column format.

- |                                      |   |
|--------------------------------------|---|
| 1. $x^2 + 3x + 2 = 0$                | Given.                                  |
| 2. $x^2 + (2 + 1)x + 2 = 0$          | In $Z$ we have $2 + 1 = 3$ .            |
| 3. $x^2 + 2x + x + 2 = 0$            | Distributive law.                       |
| 4. $(x^2 + 2x) + (x + 2) = 0$        | Associative law.                        |
| 5. $x(x + 2) + (x + 2) = 0$          | Distributive law.                       |
| 6. $x(x + 2) + 1(x + 2) = 0$         | Multiplicative identity.                |
| 7. $(x + 1)(x + 2) = 0$              | Distributive law (on the $x + 2$ term). |
| 8. Either $x + 1 = 0$ or $x + 2 = 0$ | Integral domains have no zero-divisors. |
| 9. Either $x = -1$ or $x = -2$       | Uniqueness of the additive inverse.     |

In step 2, we could have used the definition of  $3x$  in a ring in the place of the distributive law. That is, in an arbitrary ring, we usually define  $3x = x+x+x$ , and then we can use the associative law, however, I think we are better off using the distributive law when working with real numbers, because in other circumstances, you might need to break the term up into non-integer pieces. In the last step, we could have used the well-defined property for addition and have stated that  $(x + 1) + (-1) = 0 + (-1)$  (and similarly for the  $x + 2$  term. In this case, we would then need to use the associative law and the additive inverse law to get to our final answer.

On the grading of this problem, I was relatively generous. For a first time through, most people did fairly well. I would like you, however, to review the above steps so that the next time you will get even more of them.

2. *Using the axioms of a field, explain what the expression  $a$  divided by  $b$  equals  $x$  means. Why can't we divide by 0?*

At some point I will get this problem stated so that you understand more clearly what I mean. I probably shouldn't have mentioned the axioms of a field explicitly, but I want you to think on that level. In a generic field, it is not correct to write the expression  $\frac{a}{b}$ . This is a fraction. What we mean by  $a$  divided by  $b$  equals  $x$  is precisely that  $a = bx$ . Now, in a field there is a slightly different meaning. Namely,  $a$  divided by  $b$  can also mean  $ab^{-1}$ , where  $b^{-1}$  is the multiplicative inverse of  $b$  given in the field axioms. (This is what I was really looking for.) Under this meaning, division by 0 would correspond to multiplying by  $0^{-1}$ . Now, we can't divide by 0 if 0 has no multiplicative inverse. If 0 had a multiplicative inverse, then we would have a  $c \in F$  such that  $0c = 1$ . However,  $0c = (0 + 0)c = 0c + 0c$ , and adding the additive inverse of  $0c$ , we obtain  $0 = 0c$ . Thus  $0 = 0c = 1$ . In a field however, the multiplicative identity is required to be different from 0.

Under the definition that  $a = bx$ , then division by 0 means that we need to solve the equation  $a = 0x$ . However, we just showed that this equation has no solutions unless  $a = 0$ . In this case, though, any field element would be a solution, which leaves us with an ill-defined term.

Why bother with all of this? One of the classic difficulties that students have with division is upon dividing fractions. In particular, students will learn the rule "invert and multiply", but then they will be unable to apply it properly (this difficulty in applying algorithms appropriately is pretty common because students often learn algorithms without being able to give them any meaning). Consequently, they will invert the wrong fraction when they multiply, or invert incorrectly. One way to try and cure this problem when students make it is to use the associative law of multiplication and show them how they can instead multiply by the multiplicative inverse of the dividend. A second problem is that students will treat fractions symbolically and miss out on divisions by 0 when they occur (in hidden fashion). This often happens because symbolically there is no reason that you cannot divide by 0.