Solutions and Comments
Homework 4

[15/17] *Suppose a, b, and c are integers such that $\sqrt{a}+\sqrt{b}=\sqrt{c}$. Show that $\sqrt{ab}$, $\sqrt{ac}$ and $\sqrt{bc}$ are all integers. Using this, show that there exists an integer d such that $\sqrt{a}=a'\sqrt{d}$, $\sqrt{b}=b'\sqrt{d}$, and $\sqrt{c}=c'\sqrt{d}$ with a', b', and c' all integers.*

Suppose a, b, and c are as above. Then squaring both sides of the equation $\sqrt{a}+\sqrt{b}=\sqrt{c}$ yields that $a+2\sqrt{ab}+b=c$. Consequently, $2\sqrt{ab}=c-a-b$ is an integer. Moreover, $\sqrt{ab}=(c-a-b)/2$ is a rational number. Squaring again, we obtain that $ab=\frac{(c-a-b)^2}{4}$, and as ab is an integer, it follows that 4 divides $(c-a-b)^2$. Thus $(c-a-b)^2$ is even (is divisible by 2), so that Euclid's Lemma implies that 2 divides $c-a-b$, and hence $\sqrt{ab}=\frac{c-a-b}{2}$ is an integer.

To see that $\sqrt{ac}$ is an integer, we could use a similar argument on $\sqrt{b}=\sqrt{c}-\sqrt{a}$, or we could do the following based on having already done the above argument (thanks to many students on this one): Multiplying both sides of $\sqrt{a}+\sqrt{b}=\sqrt{c}$ by $\sqrt{a}$, we obtain that $\sqrt{ac}=a+\sqrt{ab}$. As a and $\sqrt{ab}$ are both integers, and thus their sum is also an integer, implying $\sqrt{ac}$ is an integer. One similarly obtains that $\sqrt{bc}$ is an integer.

For the second part, we need to actually define a', b', c', and d. Let

$$\begin{aligned} a &= p_1^{k_1}\dots p_n^{k_n}, \\ b &= p_1^{l_1}\dots p_n^{l_n}, \quad \text{and} \\ c &= p_1^{m_1}\dots p_n^{m_n} \end{aligned}$$

be the prime factorizations of a, b, and c where all the p_is are prime, and the k_is, l_is, and m_is are all non-negative integers. Define

$$\begin{aligned} a' &= p_1^{[k_1/2]}\dots p_n^{[k_n/2]}, \\ b' &= p_1^{[l_1/2]}\dots p_n^{[l_n/2]}, \quad \text{and} \\ c' &= p_1^{[m_1/2]}\dots p_n^{[m_n/2]}. \end{aligned}$$

We then define $d_a=\frac{a}{a'^2}$, $d_b=\frac{b}{b'^2}$, and $d_c=\frac{c}{c'^2}$, so that it would suffice to show that $d_a=d_b=d_c$ as we could then choose $d=d_a$. In the prime factorization of d_a, the prime p_i appears to the power $k_i-2[k_i/2]$, which is 0 if k_i is even and is 1 if k_i is odd. Similarly, in the prime factorization for

d_b, the prime p_i appears to the power $l_i - [l_i/2]$, which is 0 if l_i is even and is 1 if l_i is odd. As $\sqrt{ab}$ is an integer, it follows that ab is a perfect square so that every prime is raised to an even power. But the prime p_i is raised to the power $k_i + l_i$ in this product, which is even if and only if $k_i \equiv l_i$ modulo 2, which is to say if and only if both are even or both are odd. Consequently, p_i is raised to the same power in both d_a and d_b. As p_i was an arbitrary power from the prime factorization of these, it follows that d_a and d_b have the same prime factorization and are thus equal. A similar argument shows that $d_c = d_a$.

[17/19] There is a slightly different proof for the irrationality of π given by Ian Stewart, which we outline here. Suppose $\pi = \frac{a}{b}$ with a and b positive integers. Let

$$I_n = \int_{-1}^{+1} (1-x^2)^n \cos(\alpha x) dx.$$

1. Use integration by parts to express $\alpha^2 I_n$ in terms of $4n$, I_{n-1}, and I_{n-2}.

   Letting $u = (1-x^2)^n$ and $dv = \cos(\alpha x)dx$, integrating by parts once implies for $n \geq 1$

$$\begin{aligned} \alpha^2 I_n &= \alpha^2([(1-x^2)^n(1/\alpha)\sin(\alpha x)]_{-1}^{1} \\ &\quad -\frac{1}{\alpha}\int_{-1}^{+1} n(-2x)(1-x^2)^{n-1}\sin(\alpha x)dx) \\ &= \alpha\int_{-1}^{+1} n(2x)(1-x^2)^{n-1}\sin(\alpha x)dx). \end{aligned}$$

   Integrating this by parts with $u = x(1-x^2)$ and $dv = \sin(\alpha x)dx$, yields for $n \geq 2$ that

$$\begin{aligned} \alpha^2 I_n &= \alpha([x(1-x^2)^{n-1}(-1/\alpha)\cos(\alpha x)]_{-1}^{1} \\ &\quad -\frac{-1}{\alpha}\int_{-1}^{+1}\left(n(n-1)(2x)(-2x)(1-x^2)^{n-2} + 2n(1-x^2)^{n-1}\right)\cos(\alpha x))dx) \\ &= (-4n(n-1))\int_{-1}^{+1} x^2(1-x^2)\cos(\alpha x)dx \\ &\quad +\int_{-1}^{1} 2n(1-x^2)^{n-1}\cos(\alpha x)dx \\ &= (-4n(n-1))\int_{-1}^{+1} x^2(1-x^2)\cos(\alpha x)dx + 2nI_n. \end{aligned}$$

Now,

$$I_{n-1} - I_{n-2} = \int_{-1}^{1} (1-x^2)^{n-1}\cos(\alpha x)dx - \int_{-1}^{1}(1-x^2)^{n-2}\cos(\alpha x)dx,$$

and combining the integrals and factoring out $(1-x^2)^{n-2}\cos(\alpha x)$ out yields

$$\begin{aligned} I_{n-1} - I_{n-2} &= \int_{-1}^{1} ((1-x^2)-1))(1-x^2)^{n-2}\cos(\alpha x)dx \\ &= \int_{-1}^{1}(-x^2)(1-x^2)^{n-2}\cos(\alpha x)dx. \end{aligned}$$

Using this in our formula for $\alpha^2 I_n$, we obtain:

$$\begin{aligned} \alpha^2 I_n &= (4n(n-1))(I_{n-1} - I_{n-2}) + 2nI_n \\ &= (n(4n+2))I_{n-1} - 4n(n-1)I_{n-2}. \end{aligned}$$

2. Use induction on n to show that

$$\alpha^{2n+1} I_n = n!(P_n \sin(\alpha) + Q_n \cos(\alpha)),$$

where P and Q are polynomials in α of degree less than $2n+1$ with integer coefficients.

We will use the strong form of mathematical induction. Namely, we need to show the result for I_0 and I_1, and then show for all $n \geq 1$, that if the result is true for I_{n-1} and I_n, then the result is true for I_{n+1}.

A quick calculation shows for $n = 0$ that

$$\alpha I_0 = \int_{-1}^{1}\cos(\alpha x)dx = \sin(\alpha) - \sin(-\alpha) = 2\sin(\alpha).$$

Thus, choosing $P_0 = 2$ and $Q_0 = 0$, we have that $\alpha I_0 = 0!(P_0 \sin(\alpha) + Q_0\cos(\alpha))$, where the degree of P_0 is 0.

Using our result from the first part, we know that

$$\begin{aligned} \alpha^3 I_1 &= \alpha^2 \int_{-1}^{+1}(2x)\sin(\alpha x)dx \\ &= \alpha^2\left([(2x)(-1/\alpha)\cos(\alpha x)]_{-1}^{1} + 2(1/\alpha)\int_{-1}^{1}\cos(\alpha x)dx\right) \\ &= -4\alpha\cos(\alpha) + 2[\sin(\alpha x)]_{-1}^{1} \\ &= 4\sin(\alpha) - 4\alpha\cos(\alpha). \end{aligned}$$

Thus, choosing $P_1 = 4$ and $Q_1 = -4\alpha$, we have that the result holds for $n = 1$.

Fix $n \geq 1$, and assume that $\alpha^{2n-1} I_{n-1} = (n-1)!(P_{n-1}\sin(\alpha) + Q_{n-1}\cos(\alpha))$, where P_{n-1} and Q_{n-1} are polynomials in α of degree less than or equal to $n-1$, and assume that $\alpha^{2n+1} I_n = n!(P_n \sin(\alpha) + Q_n \cos(\alpha))$ where P_n and Q_n are polynomials in α of degree less than or equal to n. Using our recurrence relation from part one, we have

$$\begin{aligned}
\alpha^{2(n+1)+1} I_{n+1} &= \alpha^{2n+1}(\alpha^2 I_{n+1}) \\
&= (n+1)(4(n+1)+2)\alpha^{2n+1} I_n - 4(n+1)n\alpha^2\alpha^{2n-1} I_{n-2} \\
&= (4n+6)\cdot(n+1)(n!)(P_n \sin(\alpha) + Q_n \cos(\alpha)) \\
&\quad -4\alpha^2 \cdot (n+1)n((n-1)!)(P_{n-1}\sin(\alpha) + Q_{n-1}\cos(\alpha)),
\end{aligned}$$

where we used the induction hypothesis on the last step. Gathering $(n+1)!$ terms, this is equal to:

$$(n+1)!\left(\left((4n+6)P_n - 4\alpha^2 P_{n-1}\right)\sin(\alpha) + \left((4n+6)Q_n - 4\alpha^2 Q_{n-1}\right)\cos(\alpha)\right).$$

Taking $P_{n+1} = (4n+6)P_n - 4\alpha^2 P_{n-1}$ and $Q_{n+1} = (4n+6)Q_n - 4\alpha^2 Q_{n-1}$, we have that P_{n+1} and Q_{n+1} are polynomials in α of degree less than or equal to the maximum of $\deg(P_n)$, $\deg(Q_n)$, $\deg(P_{n-1}) + 2$, and $\deg(Q_{n-1} + 2)$. From the induction hypothesis, each of these is less than or equal to $n+1$, so that we have the result for $n+1$. Thus by the principle of mathematical induction, the result holds true for all n.

3. Set $\alpha = \pi/2$. Letting $J_n = a^{2n+1} I_n / n!$, show that J_n is an integer and that $0 < |J_n| \leq 2a^{2n+1}/n!$.

   The proof of this will take two steps. First we need to see that J_n is an integer, and second we shall show the inequality. As $\alpha = \pi/2$ in this case, $\cos(\alpha) = 0$ and $\sin(\alpha) = 1$. Thus, by the previous problem, $\alpha^{2n+1} I_n = n! P_n$, where P_n is a polynomial with integer coefficients of degree less than or equal to $2n+1$. Since $\alpha = a/b$, it follows that

$$J_n = a^{2n+1} I_n / n! = b^{2n+1} P_n.$$

   Now, since P_n is a polynomial in α, we can rewrite P_n as

$$P_n = d_0 + d_1\alpha + d_2\alpha^2 + \ldots + d_{2n+1}\alpha^{2n+1},$$

where the d_i are all integers. Plugging $\alpha = a/b$, we obtain that

$$b^{2n+1}P_n = b^{2n+1}(d_0 + d_1\frac{a}{b} + d_2\frac{a^2}{b^2} + \ldots + d_{2n+1}\frac{a^{2n+1}}{b^{2n+1}}).$$

Distributing the b^{2n+1} term throughout, we obtain that

$$J_n = d_0b^{2n+1} + d_1ab^{2n} + \ldots + d_{2n+1}a^{2n+1},$$

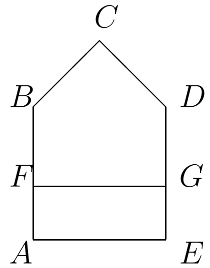
which is an integer as every term is an integer. Thus J_n is an integer. Since $I_n = \int_{-1}^{+1}(1-x^2)^n\cos(\alpha x)dx$, and $0 < (1-x^2)^n\cos(\alpha x) \leq 1$ for all $x \in (-1,1)$, it follows that $0 < I_n \leq 2$. Thus,

$$0 < J_n \leq \frac{2a^{2n+1}}{n!}.$$

4. Use the above step to establish a contradiction, so that π must be irrational.

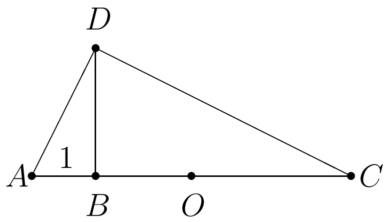
   Since a is a fixed integer, $2a^{2n+1} < n!$ for some n sufficiently large. Thus, $0 < J_n < 1$ for n sufficiently large. But J_n is an integer, so this is impossible. Thus our assumption that $\pi = a/b$ for some integers a and b must have been false. Hence π is irrational. (Whew!)

[Pentagon:] Give an example of two pentagons which have corresponding angles congruent but are not similar.



The pentagons $ABCDE$ and $FBCDG$ have congruent angles, but are not similar. Given any pentagon, you can quickly construct an example, by simply inserting a parallel line inside of one side. Actually, you can use this technique to construct two pentagons that are completely dissimilar in the sense that no pair of sides is in the same ratio.

[**Square Root Construction**] Given the following picture, prove that $|\overline{DB}| = \sqrt{|\overline{CB}|}$, given that the circle centered at O through A, also runs through C and D.



This proof is thanks to Tim Fleming. To start with, we need to show that $\angle ADC$ is a right angle. Note that we are given that DB is perpendicular to AC. Draw in the auxiliary line $\overline{OD}$. It then follows that triangles ODC and OAD are isosceles. Thus $\angle ODC \cong \angle OCD$ and $\angle ODA \cong \angle OAD$. Clearly, $\angle ADC$ is the sum of the two angles at D, and the congruences then tell us that the measure of $\angle ADC$ is the sum of measures of $\angle OAD$ and $\angle OCD$. We will write this measure as $m(\angle ADC)$. As the sum of the angles of a triangle is π radians or $180°$, it follows that

$$\begin{aligned} \pi &= m(\angle ADC) + m(\angle OAD) + m(\angle(OCD) \\ &= m(\angle ADC) + m(\angle ADC) = 2m(\angle ADC). \end{aligned}$$

Consequently, $m(\angle ADC) = \pi/2$, and it is a right angle. As $\angle ACD$ is an angle of both triangle ACD and DCB and both of these triangles contain a right angle, the AA similarity theorem implies $\triangle ACD \sim \triangle DCB$. Similarly (no pun intended), as $\angle CAD$ is an angle of both $\triangle ACD$ and $\triangle ADB$ and both contain a right angle, we have $\triangle ACD \sim \triangle ADB$. As similarity is an equivalence relation(!), it follows that $\triangle DCB \sim triangADB$. Using the proportionality of corresponding sides, we obtain:

$$\frac{|\overline{AB}|}{|\overline{BD}|} = \frac{|\overline{BD}|}{|\overline{BC}|}.$$

Solving this for $|\overline{BD}|$, we obtain that

$$|\overline{BD}| = \sqrt{|\overline{BC}|}$$

as desired.

[**Worksheet problems:**] I am not going to give full scale solutions to these problems as everyone got problem 1 and problem 2, problem 5 will be discussed on the next solution set, and problem 3 everyone had except that they didn't necessarily give the reasons for the triangles being similar in the proper way. For the construction of the product, note that this can be done with any triangle, (not necessarily a right triangle) as long as you can copy an angle. The idea is as follows.

Draw a line l.
Mark a point A on l.
Mark a point B on l at distance 1 from A.
Mark a point C on l at distance a from A so that B lies on $\overline{AC}$.
Draw a circle of radius b centered at B.
Mark a point on the circle not on l. We now
Draw a line m through A and D.
Let n be the line segment BD.
Copy the angle ABD at point C (as done in class).
Let l' be the line making this angle with l.
Mark the intersection point E of m and l'.

claim that $|\overline{CE}| = ab$. To see this, note that $\angle BAD \cong \angle CAE$ since they are the same angle, and that $\angle ABD \cong \angle ACE$ by construction. Thus $\triangle ABD \cong \triangle ACE$. Using similar triangles, it follows that $\frac{|\overline{AB}|}{|\overline{BD}|} = \frac{\overline{AC}}{|\overline{CE}|}$, or writing $x = |\overline{CE}|$, we have $\frac{1}{a} = \frac{b}{x}$, so that $x = ab$.

To construct the quotient, you follow the steps:

Draw a line l.
Mark a point A on l.
Mark a point B on l at distance b from A.
Mark a point C on l at distance a from A so that B lies on the ray from A to C.
Draw a circle of radius 1 centered at B.
Mark a point on the circle not on l.
Draw a line m through A and D.
Let n be the line segment BD.
Copy the angle ABD at point C (as done in class).
Let l' be the line making this angle with l.
Mark the intersection point E of m and l'.

Again, we have similar triangles and we now get the relation $\frac{1}{b} = \frac{\overline{CE}}{a}$. Consequently, $\overline{CE} = a/b$.