Homework 11

Due Wednesday November 29, 2000

Please start before next week!

1. Prove that the sum of two Dedekind cuts (A, B) and (C, D) (where in the case that either (or both) is a rational cut, we only consider the cut with the corresponding rational number on the right) is a Dedekind cut. (Recall that (A, B) is a Dedekind cut if (i) $A \neq \emptyset, B \neq \emptyset$, (ii) $A \cup B = Q$, (iii) $A \cap B = \emptyset$, and (iv) if $a \in A$ and $b \in B$ then $a < b$, and the *sum* of two Dedekind cuts (A, B) and (C, D) is defined to be the pair (E, F), where $E = \{a + c \mid a \in A, c \in C\}$ and $F = \{b + d \mid b \in B, d \in D\}$.)

2. My five function calculator at home caries exactly eight digits. To find the decimal equivalent to $\sqrt{2} + \sqrt{3}$, I add the value the calculator gives me for the $\sqrt{2}$ and the value it gives me for $\sqrt{3}$ to get 3.1462643. How accurate is this? To find the value of $\frac{1}{9} \cdot 9$, I divide 1 by 9 and then multiply by 9, at which point the calculator gives me 0.9999999. How accurate is this? In general, if I add two numbers on this calculator, how inaccurate can the answer be? (For this problem, give both the number of digits you can be certain of and the percentage error (*i,e.*, if δ is the error and C is the correct answer, what is the maximum value of δ/C?)

3. In the previous problem, I added numbers. What is the worst possible error that can arise from *multiplying* two numbers on this calculator? Do this both in terms of the number of digits, and in terms of the percentage error.

4. In class, we defined the infinite decimal $\sum_{i=0}^{\infty} a_i 10^{-i}$ as the number corresponding to the cut (A, B), where

$$B = \{q \in Q \mid \text{for all } k \in N,\ q > sum_{i=0}^{k} a_i 10^{-i}\ \}$$

and

$$A = \{q \in Q \mid \text{there exists } k \in N \text{ such that } q \leq sum_{i=0}^{k} a_i 10^{-i}\ \}.$$

Moving in the opposite direction to each real number (*i.e.,* Dedekind Cut (A, B)), we can associate a sequence of *terminating* decimal expressions (and hence rational numbers!!), $q_0, q_1, \ldots$ such that (A, B) is the least upper bound of $\{q_i \mid i \in N\}$, and having the property that $q_i - q_{i+1} \leq 10^{-i}$.

Relate the above two problems to the addition and multiplication of Dedekind cuts and their infinite decimals.

5. Suppose we have k real numbers $a_1, \ldots, a_k$ each (strictly) between 1 and 10 given by their eight digit approximation on my calculator. How accurate is the calculator in giving their product? Explain.

6. Left to their own devices, some children will develop an addition algorithm which starts by adding the left-most term first and then moving successively to the right. Discuss the relative merits of this algorithm as compared to the standard algorithm for terminating decimals. What about for non-terminating decimals? How do the above problems relate to this question?