Homework 10

Due, Wednesday, November 22

So far we have talked about rational numbers (and their decimal equivalents), constructible numbers, algebraic numbers, and transcendental numbers. We have now completed the set of real numbers.

1. Sketch a diagram relating the natural numbers, the integers, the rational numbers, the constructible numbers, the algebraic numbers, and the transcendental numbers. Explain the linkages and include, if necessary, other sets of numbers needed to fill out the picture.

2. The guiding principle in defining the real numbers is the number line. What are all the properties that the set of real numbers should have? You should come up with at least twelve properties that are expected of the set (Hint: Think about the entire field).

   After trying to come up with answers to this question, pick up the pink sheet and see how many of the needed properties you have identified. On that pink sheet, identify (in words) the ramification or meaning of each axiom.

3. Using the axioms on the pink sheet, let us move on to prove some things that we expect about the real numbers.

   (a) If $x < y$, then $-y < -x$. Hint: Assume $x < y$ and add the same thing to both sides of the inequality in order to establish that $-y < -x$.

   (b) $0 < 1$. Although this statement seems obvious, you need to establish this result using the pink sheets axioms. What are the three possible ways of relating 0 and 1? Use the axioms to establish that what you know is impossible is truly impossible.

(c) If $0 < x < y$, then $0 < \frac{1}{x} < \frac{1}{y}$.

i. Assume $0 < x < y$. What are the possibilities for the relationship between 0 and $\frac{1}{x}$?

ii. Argue why, from the set of axioms, two of these possibilities cannot be true.

iii. What does this mean about $\frac{1}{y}$?

iv. Using $0 < x < y$, multiply each element of the compound inequality by the same factor in order to establish $0 < \frac{1}{y} < \frac{1}{x}$. Explain why this is legal.

(d) If $x < y$ and $z < 0$, then $yz < xz$. Hint: Assume $x < y$ and $z < 0$. Use axiom 11 and a previous result to establish $yz < xz$.

4. Prove Theorem 4.1: If S is a non-empty set of real numbers that is bounded from above, then S has a least upper bound.

   (a) Argue why if M is an upper bound for S, then $M \geq s$ for all $s \in S$.

   (b) Let $T = \{t \mid t = -s, \text{for some } s \in S\}$. Argue why $-M$ is a lower bound for T.

   (c) Explain why if K is a lower bound for T, then $-K$ is an upper bound for S.

   (d) Argue why T must have a greatest lower bound, call it B.

   (e) Argue why $-B$ must be an upper bound for S.

   (f) If C is an upper bound for S, then what is the relationship between $-C$ and B as well as $-C$ with the set T?

   (g) What is the relationship between C and $-B$? How does this imply $-B$ is the least upper bound for S?

5. Prove Theorem 4.2: Let x be any real number. Then there is an integer $n \leq x < n+1$.
In order to accomplish this, we need to define the set
$A = \{n \mid n \in Z \text{ and } n \leq x\}$.

   (a) If A is non-empty, argue why A is bounded from above and has a least upper bound (call it n_0).

   (b) Explain why if you subtract 1 from n_0, the resultant value will not be an upper bound of A.

   (c) Discuss why there exists an $m \in A$ such that $n_0 - 1 < m \leq n_0$.

   (d) Explain why $m + 1 \notin A$ and $x < m + 1$.

   (e) How does this establish, for the case that A is non-empty, the result that we can find an integer n such that $n \leq x < n + 1$.

   (f) To show that A is not empty, consider the set $B = \{b \mid b > x \text{ and } b \in Z\}$, and show that it must have a least upper bound called b_0. Argue that there exists a $m \in B$ such that $b_0 \leq m < b_0 + 1$ so that $m - 1 \in A$.

6. Prove Theorem 4.3: Between any two real numbers is both a rational number and an irrational number.

   (a) Let x and y be real numbers with $x < y$. Why is there an integer N such that $0 < \frac{2}{y-x} < N$?

   (b) Argue why there exists an integer $n \leq Nx < n + 1$.

   (c) Argue why $n + 2 < Ny$.

   (d) Explain why $\frac{n+1}{N}$ and $\frac{n+2}{N}$ are rational numbers between x and y.

   (e) Explain why $\frac{n+\sqrt{2}}{N}$ is an irrational number between x and y.

7. Take a look at the Numbers and Operations strand from the Principles and Standards for School Mathematics: Discussion Draft (NCTM 1998) attached to this, or go to the web site www.standards.nctm.org. Address how they built the idea of number throughout the grade levels by answering the following questions:

   (a) What is the sequence of development of the number concept as evidenced in the Standards?

   (b) Do the Standards suggest that the students be introduced to the properties of the real numbers? In your opinion, what would be the rationale for doing this? How could a teacher coherently present these ideas?

   (c) In the 9-12 standards, matrices and complex numbers are discussed. What is the purpose for inclusion of these discussions? Explain how you arrived at this conclusion and what evidence you have to support your opinion.