

Final Study Guide
MATH 111

Sample Problems on
Algebra, Functions, Exponents, & Logarithms

Math 111

Part 1: No calculator or study sheet. Remember to get full credit, you must show your work.

1. Determine graphically the solution set for each system of inequalities and indicate whether the solution set is bounded or unbounded:

$$\begin{aligned}x - y &< 7 \\3x + 2y &\geq 6 \\x &\geq 0\end{aligned}$$

Solution: The solution is unbounded. To solve it graphically, you need to graph the three lines: $x - y = 7$ (which passes through $(0, -7)$ and $(7, 0)$, graphed with a dashed line), the line $3x + 2y = 6$ (which passes through $(0, 3)$ and $(2, 0)$, and the line $x = 0$ (the y -axis). After checking a test point, we see that the solution is the shaded region to the upper right of the triangle. including the two solid lines as boundary. (I will try and get a picture solution on the web soon, but I am having trouble getting one converted to the type of file I need to do so.)

2. Suppose we are given a linear programming problem with a feasible set S and an objective function $P = ax + by$. Finish the following sentences:
 - (a) If S is _____ then P has both a maximum and a minimum value on S .
 - (b) If S is _____ and both a and b are nonnegative, then P has a minimum value on S provided that the constraints defining S include the inequalities $x \geq 0$ and $y \geq 0$.
 - (c) If S is _____ then the linear programming problem has no solution; that is P has neither a maximum nor a minimum value.

Solution: These are from a theorem in the book.

- (a) If S is bounded then P has both a maximum and a minimum value on S .
 - (b) If S is unbounded and both a and b are nonnegative, then P has a minimum value on S provided that the constraints defining S include the inequalities $x \geq 0$ and $y \geq 0$.
 - (c) If S is the empty set then the linear programming problem has no solution; that is P has neither a maximum nor a minimum value.
3. Give an example of a simplex tableau that is in final form and one that is not in final form.

Solution: A simplex in final form is

$$\left[\begin{array}{cccccc} x & y & u & v & P & \\ 0 & 1 & 6 & 2 & 0 & 10 \\ 1 & 0 & -4 & -3 & 0 & 15 \\ \hline 0 & 0 & 8 & 4 & 1 & 100 \end{array} \right]$$

A simplex that is not in final form is

$$\left[\begin{array}{cccccc} x & y & u & v & P & \\ 1 & 3 & 1 & 0 & 0 & 10 \\ 1 & 2 & 0 & 1 & 0 & 15 \\ \hline -2 & -3 & 0 & 0 & 1 & 100 \end{array} \right]$$

The latter corresponds to the optimization problem

$$\begin{aligned} P &= 2x + 3y \\ x + 3y &\leq 10 \\ x + 2y &\leq 15 \\ x, y &\geq 0 \end{aligned}$$

4. Suppose that you take out a loan for \$30,000 over a 10 year term at 5% interest with monthly payments. At the end of 3 years do you anticipate that you would owe closer to \$15,000, \$17,500, \$20,000, or \$22,500. Explain. (You should think about how to solve this without a calculator.)

Solution: If there were no interest on the loan, then in 3 years we would pay off $3/10$ of the loan value. That is \$9,000. Thus, we must have at least \$21,000 left on the principle of the loan. Since much of our payments would be towards interest early on, we would anticipate that the actually amount we owe would be closer to \$22,500.

5. True or false: the future value of an annuity can be found by adding together all the payments that are paid into the account.

6. State the quadratic formula.

Solution: If $ax^2 + bx + c = 0$ with $a \neq 0$, and a , b , and c constants then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

7. Factor out the greatest common factor from each expression

(a) $3x^5 - 12x^3 + 9x^2$

Solution: $3x^2(x^3 - 4x + 3)$.

(b) $4x^{2/3}y - 3xy^{1/3}$.

Solution: $x^{2/3}y^{1/3}(4y^{2/3} - 3x^{1/3})$.

8. Factor $9a^2 - x^2$.

Solution: $9a^2 - x^2 = (3a - x)(3a + x)$.

9. Given that 2 is a real root of the polynomial $f(x)$, state a factor of $f(x)$.

Solution: By the factor theorem $x - 2$ is a factor of $f(x)$.

10. Evaluate the given expression

(a) $|-5 + 3|$

Solution: $|-5 + 3| = 2$.

(b) $\sqrt{3} \cdot |-4| + 2 \cdot |-\sqrt{3}|$.

Solution: $\sqrt{3} \cdot |-4| + 2 \cdot |-\sqrt{3}| = 4\sqrt{3} + 2\sqrt{3} = 6\sqrt{3}$.

11. Simplify the following expressions:

(a) $\frac{x^{-1}+2x^{-2}}{x^2-4}$

Solution: We can add the fractions in the numerator and then turn this into a simple fraction.

$$\begin{aligned}\frac{x^{-1} + 2x^{-2}}{x^2 - 4} &= \frac{\frac{1}{x} + \frac{2}{x^2}}{x^2 - 4} \\ &= \frac{\frac{x}{x^2} + \frac{2}{x^2}}{x^2 - 4} \\ &= \frac{\frac{x+2}{x^2}}{x^2 - 4} \\ &= \frac{x+2}{x^2} \cdot \frac{1}{x^2 - 4} \\ &= \frac{x+2}{x^2} \cdot \frac{1}{(x-2)(x+2)} \\ &= \frac{1}{x^2(x-2)}.\end{aligned}$$

(b) $\frac{-xe^x}{x+1} + e^x$

Solution: We find a common denominator and simplify

$$\begin{aligned}\frac{-xe^x}{x+1} + e^x &= \frac{-xe^x}{x+1} + \frac{e^x(x+1)}{x+1} \\ &= \frac{-xe^x + e^x(x+1)}{x+1} \\ &= \frac{e^x}{x+1}.\end{aligned}$$

(c) $|\sqrt{2} - \sqrt{6}|$.

Solution: Since $\sqrt{6} > \sqrt{2}$, then $\sqrt{2} - \sqrt{6} < 0$. Thus

$$|\sqrt{2} - \sqrt{6}| = -(\sqrt{2} - \sqrt{6}) = \sqrt{6} - \sqrt{2}.$$

12. Find the values of x that satisfy the inequality (inequalities):

(a) $-3x \leq 24$.

Solution: Dividing both sides by -3 , we obtain: $x \geq -8$. In interval notation the solution is $[-8, \infty)$.

(b) $x^2 - 5x + 5 > 1$.

Solution: First we subtract 1 from each side, yielding.

$$\begin{aligned}x^2 - 5x + 4 &> 0 \\(x - 4)(x - 1) &> 0.\end{aligned}$$

Now we look at where we have equality, namely at $x = 4$ and $x = 1$, and we can check whether the inequality holds for $x < 1$, $1 < x < 4$, and $4 < x$. Checking test points $x = 0$, $x = 2$, and $x = 5$. For $x = 0$, the inequality is $4 > 0$, which is true. Thus for $x < 1$ the inequality holds. If $x = 2$, then we have $-2 > 0$ which is incorrect, so the inequality does not hold for $1 < x < 4$. Finally, if $x > 5$, the inequality becomes $4 > 0$, which is correct. Thus we have that the inequality holds if

$$x < 1 \text{ or } x > 4.$$

In interval notation, $(-\infty, 1] \cup [4, \infty)$.

(c) $|x - 4| \leq 9$.

Solution: We can interpret this as “ x is within 9 of 4”, meaning that

$$-5 \leq x \leq 13,$$

or in interval notation $[-5, 13]$.

Alternative Solution: Using the more standard method, $|x - 4| \leq 9$ is equivalent to

$$-9 \leq x - 4 \leq 9.$$

Adding 4 to each side, we obtain

$$-5 \leq x \leq 13.$$

(d) $|3x + 7| \geq 5$.

Solution: Rewriting this as $|3x - (-7)| \geq 5$, this inequality can be interpreted as “ $3x$ is more than 5 away from -7 .” To be more than 5 away from -7 , we have that $3x$ should be less than or equal to -12 or $3x$ should be greater than -2 . Thus $3x \leq -12$ or $3x \geq -2$. Dividing by 3 we have $x \leq -4$ or $x \geq -2/3$. In interval

notation, $(-\infty, -4] \cup [-2/3, \infty)$.

Alternative Solution: The more traditional solution is that $|3x + 7| \geq 5$ corresponds to one of two possibilities (1) $3x + 7 \geq 5$ or (2) $3x + 7 \leq -5$.

We first deal with (1). Subtracting 7 from each side we obtain

$$3x \geq -2,$$

and dividing by 3 we get $x \geq -2/3$.

In the case of (2) we have

$$3x \leq -12$$

after we subtract 7 from both sides and then dividing by 3 we have

$$x \leq -4.$$

Thus we have $x \leq -4$ or $x \geq -2/3$.

13. Evaluate the following:

(a) $(64)^{4/3}$.

Solution: This was supposed to read $(64)^{4/3}$. Factoring 64, we have $64 = 2^6$, so

$$\begin{aligned} 64^{4/3} &= (2^6)^{4/3} \\ &= 2^{6 \cdot 4/3} \\ &= 2^8 \\ &= 256. \end{aligned}$$

As written the problem is a little more painful since $64^4 = 2^{24}$ which is pretty humongous. Sorry.

(b) $\left[\left(\frac{-1}{3}\right)^2\right]^{-3}$.

Solution: To solve this,

$$\begin{aligned} \left[\left(\frac{-1}{3}\right)^2\right]^{-3} &= \left[\frac{-1}{3} \cdot \frac{-1}{3}\right]^{-3} \\ &= \left[\frac{1}{9}\right]^{-3} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9^{-3}} \\
&= 9^3 \\
&= 729.
\end{aligned}$$

14. Simplify

(a) $(2x^3)(-4x^{-2})$.

Solution: $(2x^3)(-4x^{-2}) = -8x$.

(b) $\frac{e^x e^{4x}}{e^{2x-2}}$.

Solution:

$$\begin{aligned}
\frac{e^x e^{4x}}{e^{2x-2}} &= \frac{e^{5x}}{e^{2x-2}} \\
&= e^{5x-(2x-2)} \\
&= e^{3x+2}.
\end{aligned}$$

15. Solve for x : $10^{2x-1} = 10^{x+3}$.

Solution: Since the base is the same, we have that

$$2x - 1 = x + 3.$$

Solving the above for x we get

$$x = 4.$$

16. Use the logarithm rules to simplify the given equations: (it should say expressions)

(a) $\ln(x(x+1)(x+2))$.

Solution: $\ln(x) + \ln(x+1) + \ln(x+2)$.

(b) $\ln(x^{x^2+1})$.

Solution: $(x^2 + 1) \cdot \ln(x)$.

17. Express the given equation in logarithm form.

(a) $5^{-3} = \frac{1}{125}$.

Solution: $\log_5\left(\frac{1}{125}\right) = -3$.

(b) $e^{5x} = 72$.

Solution: $\ln(72) = 5x$.

Calculator and Study Sheet Allowed

1. Graph the functions on the same axis:

$$y = e^{3x}$$
$$y = \ln(3x).$$

Solution: If you use a graphing calculator, the upper graph is $y = e^{3x}$.

2. Graph the functions on the same axis: $y = 2^x$, $y = 3^x$, and $y = 4^x$. Clearly label each.

Solution: On a graphing calculator the lowest curve (to the right of the y -axis) is $y = 2^x$, the central curve is $y = 3^x$, and the highest curve is $y = 4^x$.

3. The revenue of Leisure Travel is given by $f(x)$ dollars, where x is the dollar amount spent by the company on advertising. The amount spent by Leisure at time t on advertising is given by $g(t)$ dollars. What does the function $f \circ g$ represent?

Solution: $f \circ g$ is the function that tells you the revenue of Leisure Travel at a given time. That is $f(g(t))$ is the revenue of Leisure Travel at time t .

4. Investment A offers a 10% return compounded semiannually, and investment B offers a 9.75% return compounded continuously. Which investment has a higher rate of return over a 4-yr period?

Solution: Over 4 years, the return on investment A is given by $(1 + .1/2)^8 = 1.47745\dots$

Based on our continuously compounding interest formula, the return on investment B is $e^{.0975 \cdot 4} = 1.4769\dots$. Thus the higher return is in investment A .

5. Find the interest rate needed for an investment of \$5000 to grow to the amount of \$7500 in 3 years if interest is compounded monthly.

Solution: For this problem, we use a future-present value equation.

We have $P = 5000$, $A = 7500$, $t = 3$, $m = 12$ and r is what we are trying to find. Thus we have the equation

$$7500 = 5000(1 + r/12)^{3 \cdot 12}.$$

Dividing both sides by 5000 we have

$$1.5 = (1 + r/12)^{36}.$$

Taking the natural logarithm of both sides, we have

$$\ln(1.5) = \ln((1 + r/12)^{36}).$$

Using the logarithm rules we have

$$\ln(1.5) = 36 \ln(1 + r/12).$$

Solving this for r

$$\begin{aligned} \frac{\ln(1.5)}{36} &= \ln(1 + r/12) \\ e^{(1/36) \ln(1.5)} &= e^{\ln(1+r/12)} \\ e^{(1/36) \ln(1.5)} &= 1 + r/12 \\ e^{(1/36) \ln(1.5)} - 1 &= r/12 \\ 12(e^{(1/36) \ln(1.5)} - 1) &= r. \end{aligned}$$

Using a calculator, we get that $r = .135919\dots$. Thus we would need to invest the money in an account receiving about 13.6% interest (compounding monthly) to get 7500 in 3 years. This makes sense since at that interest rate, we would expect the account to double in roughly $70/13.6 = 5$ years.

6. How long will it take an investment of \$4000 to triple in 5 years if the investment earns interest at the rate of 8% compounded quarterly. Approximately how long will it take to double?

Solution: There should not have been a time frame in the original problem. The correction is above. In this case, we have $P = 4000$, $r = .08$, $m = 4$, and $A = 12000$ (it is tripling), and we want to know t . Thus we have the equation

$$12000 = 4000(1 + .08/4)^{4t}.$$

Dividing by 4000 and solving below we have

$$\begin{aligned}3 &= (1 + .08/4)^{4t} \\ \ln(3) &= \ln((1 + .08/4)^{4t}) \\ \ln(3) &= 4t \ln(1.02) \\ \frac{\ln(3)}{4 \ln(1.02)} &= t.\end{aligned}$$

Evaluating the left hand side on the calculator, we have $t = 13.869\dots$. Thus it takes about 13.87 years, or since we are compounding quarterly (and interest is paid quarterly), the account would first more than triple at 14 years.

For doubling, we solve for a final value of \$8000 instead of \$12000. Following the same solution method, we have

$$t = \frac{\ln(2)}{4 \ln(1.02)} = 8.7506\dots$$

Thus the account doubles in about 8.75 years. The rule of 70 tells us we should expect the account to double in about $70/8$, which is approximately $72/8 = 9$ years, so this answer seems reasonable.

7. Suppose you take out a home loan of \$400,000 at 7% interest amortized over 30 years. The mortgage company allows you to pay no interest for the first year of the loan. At the end of the first year, the loan converts to a traditional mortgage (with only 29 years left). How much are the payments at the end of the first year? How does this compare to what the payment would be if you had taken a traditional 6.9% 30 year loan (with monthly payments).

Solution: We will be assuming monthly payments throughout.

For this problem, we first calculate what the \$400,000 would be worth in 30 years. This gives us a future value of

$$A = 400000(1 + .07/12)^{12*30} = 3246598.99.$$

In the first year of the loan, there are no payments made. Thus, we have 29 years left to pay off the loan, so we need an annuity payment

for a 29 year annuity giving a final value of 3246598.99. Thus we are solving the equation

$$3246598.99 = R \left(\frac{(1 + .07/12)^{12 \cdot 29} - 1}{.07/12} \right)$$

for R . Solving we obtain $R = 2882.874\dots$, or a monthly payment of \$2,882.88.

If we had taken out a traditional 6.9% mortgage instead, the future value would have been

$$400000 * (1 + .069/12)^{30 \cdot 12} = 3151191.809\dots$$

Now we solve for the payment in

$$3151191.809 = R \left(\frac{(1 + .069/12)^{12 \cdot 30} - 1}{.069/12} \right).$$

Solving for R we get

$$R = 2634.400\dots$$

Or monthly payments of about \$2634.41 (you always have to round up to be sure that there aren't a few pennies (or dollars) left over at the end.

Note that by not paying money the first year, you save $12 \cdot 2634.41 = 31,612.81$ dollars, but at the cost of paying roughly \$250 a month more for the next 29 years, which translates to about \$30000 a year more. This is where the risk is in taking the strange mortgages where you don't pay much the first year. The negative amortization (that is owing more at the end of a year) can kill you in terms of how much more your monthly payments will be. Something similar often can happen with credit card debt.

8. Suppose tuition at Georgetown is increasing at 12% per year, and currently costs \$20,000 per year. How much will tuition at Georgetown be in 5 years?

Solution: The future value of \$20,000 in 5 years from now will be (from our future/present value conversion formula) is

$$A = 20000(1 + .12)^5 = 35246.83\dots$$

Thus the future value will be roughly \$35,246.83.

9. A T-shirt company wants to manufacture 2 types of T-shirts. The first T-shirt requires 10 minutes on machine *A*, 5 minutes on machine *B*, and 3 minutes on machine *C*. The second T-shirt requires 7 minutes of manufacturing time on machine *A*, 6 minutes of manufacturing time on machine *B*, and 4 minutes of manufacturing time on machine *C*. The first T-shirt sells for a profit of 5 dollars and the second T-shirt sells for a profit of 6 dollars. Set up the linear programming problem for this company. Label all of your variables.

Correction to question: Oops, I didn't include the amount of time on each machine... sorry. Suppose machine *A* has 2 hours available, machine *B* has 1 hour available, and machine *C* has 3 hours available.

Solution: Let x denote the number of the first T-shirt to be made, and y denote the number of the second T-shirt to be made. For the requirements on machine *A*, we know

$$10x + 7y$$

For machine *B* we have $5x + 6y$ and for machine *C*, we have $3x + 4y$. Of course since there aren't any negative numbers of T-shirts, $x \geq 0$ and $y \geq 0$. We are trying to maximize the profit function $P = 5x + 6y$. Thus the final set of equations is:

$$\begin{aligned} P &= 5x + 6y \\ 10x + 7y &\leq 120 \\ 5x + 6y &\leq 60 \\ 3x + 4y &\leq 180. \end{aligned}$$

10. Set up a simplex tableau associated to the programming problem

$$\begin{aligned} P &= 4x + 5y + 2z \\ 3x + 5y - z &\leq 100 \\ 2x + 7z &\leq 125 \\ 4y + z &\leq 110 \\ x, y, z &\geq 0 \end{aligned}$$

Set up a simplex tableau for this problem.

Solution: We have slack variables u , v , and w for the three equations.

Then we have

$$\left[\begin{array}{cccccc|c} x & y & z & u & v & w & P \\ 3 & 5 & -1 & 1 & 0 & 0 & 100 \\ 2 & 0 & 7 & 0 & 1 & 0 & 125 \\ 0 & 4 & 1 & 0 & 0 & 1 & 110 \\ \hline -4 & -5 & -2 & 0 & 0 & 0 & 1 \\ \hline \end{array} \right]$$

11. Suppose a linear programming problem has the following set up: x denotes the number of widgets to be produced, y denotes the number of thingymabobs to be produced, and z denotes the number of whatchamacallits to be produced. There are 3 machines that each product requires time on. Suppose the final tableau for this problem is:

$$\left[\begin{array}{cccccc|c} x & y & z & u & v & w & P \\ 0 & 1 & 0 & 6 & 0 & 2 & 10 \\ 0 & 0 & 2 & 3 & 1 & 8 & 20 \\ 1 & 0 & -3 & -4 & 0 & -3 & 15 \\ \hline 0 & 0 & 3 & 8 & 0 & 4 & 1 \\ \hline \end{array} \right]$$

What is the final solution associated with this tableau? Interpret this solution using words. Does any machine have slack time?

Solution: The simplex corresponds to the following equations.

$$\begin{aligned} x &= 15 + 3z + 4u + 3w \\ y &= 10 - 6u - 2z \\ v &= 20 - 2z - 3u - 8w \\ P &= 100 - 3z - 8u - 4w \end{aligned}$$

P is maximized when $z = u = w = 0$. Thus a solution is given by: $x = 15$, $y = 10$, and $z = 0$. That is that we should make 15 widgets, 10 thingymabods, and 0 whatchamacallits.

There is slack on the machine associated to the variable v .

12. Determine the monthly payment that would be made on a 5 year car loan for \$40,000 car at 7.5% annual interest payable every month.

Solution: We first find out the final value of 40000 at 7.5% interest using the present/future value formula:

$$A = 40000(1 + .075/12)^{5 \cdot 12} = 58131.78.$$

Using the future value of the annuity formula, we have

$$58131.78 = R \frac{(1 + .075/12)^{5 \cdot 12} - 1}{.075/12} = R72.527 \dots$$

Solving for R we obtain

$$R = 801.52$$

dollars per month.

As a quick check of the sensibility of this answer is that in just pure dollar value, we pay in about 48901. Since this is more than the initial value, I think the number sounds reasonable.

13. Price publishing sells encyclopedias under two payment plans: cash or installment. Under the installment plan, the customer pays \$22 per month over a 3-year period with interest charged on the balance at a rate of 18% per year compounded monthly. Find the cash price for a set of encyclopedias if it is equivalent to the price paid by a customer using the installment plan.

Solution: An installment plan is the same any purchase over time. Thus, we need the present value of the \$22/month installment. To find the final value of the encyclopedia set is given by our annuity formula

$$S = 22 \left(\frac{(1 + .18/12)^{36} - 1}{.18/12} \right) = 1040.07$$

Now we need to find the present value using our future/present value conversion.

$$1040.07 = P(1 + .18/12)^{36}.$$

Calculating the $(1 + .18/12)^{36} = 1.70914$. Thus $P = 608.54$ dollars is the initial value of the annuity.

Thus the cash price should be 608.54