# Study Guide 1
# Solutions

# Sample Problems for Test 1
# Math 111
# Part 1: No calculator or study sheet

1. Find the slope of a line through the points $(2, -3)$ and $(5, 1)$.
   **Solution:** Slope is given by the difference of the $y$-coordinates divided by the difference of the $x$-coordinates, so the slope is given by
   $$m = \frac{1-(-3)}{5-2} = \frac{4}{3}.$$

2. List all possibilities for the number of solutions that a system of 3 equations in 3 unknowns can have. Does this answer change (and if so how) if we have 3 equations in 4 unknowns?
   **Solution:** If the number of equations is greater than or equal to the number of unknowns (as is the case for 3 equations and 3 unknowns) the possibilities are:
   - 0 solutions,
   - Exactly 1 solution, or
   - Infinitely many solutions.

   In the latter case, there are more unknowns than equations, so there are only two possibilities:
   - 0 solutions, or
   - infinitely many solutions.

3. Write the augmented matrix associated to the equations

$$\begin{aligned} 2x + 4y - z &= 1 \\ 3x - 2y + z &= 2 \\ 5x - y &= -2 \end{aligned}$$

**Solution:** $\left[\begin{array}{ccc|c} 2 & 4 & -1 & 1 \\ 3 & -2 & 1 & 2 \\ 5 & -1 & 0 & -2 \end{array}\right]$

4. If a graph has a negative slope, as you move to the right on the graph, do the $y$-values get larger or smaller?
**Solution:** As the $x$-values get larger, the $y$-values get smaller. Since the $x$-values get larger as you move to the right, the $y$-values get smaller.

5. If

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

is the reduced row echelon form of the matrix $A$ associated to a system of equations (in $x$, $y$, and $z$), write the solution of the corresponding system of equations.
**Solution:**

$$\begin{aligned} x - 2z &= 5 \\ y + z &= 2 \\ 0 &= 0 \end{aligned}$$

Do remember to include the third equation.

6. What are the three different forms for the equation of a line.
**Solution:** We actually gave 4 in class, and I would accept any 3 of the 4. They are

- Slope-intercept: $y = mx + b$.
- Point-slope: $y - y_1 = m(x - x_1)$.
- General form: $Ax + By + C = 0$ (or alternatively $Ax + By = C$ I would accept one or the other, but they count as the same form).

- Intercept Form: $\frac{x}{A} + \frac{y}{B} = 1$ (where $A$ and $B$ are the $x$- and $y$-intercepts).

I would also accept the two-point version of a line, but I would prefer the one's above.

7. Write down a matrix in row-reduced form for which the corresponding system of equations has **NO** solutions.
**Solution:** The easiest system would have only one equation, but to be slightly interesting:

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

8. Write down a matrix in row-reduced form for which the corresponding system of equations has infinitely many solutions.
**Solution:** We need a column without a leading 1. So for example:

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

**Solution:** Using that $A^{-1}A = I_3$ the $3 \times 3$ identity matrix, then

$$v + A^{-1}Av = v + v = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix},$$

9. Are the two matrices below inverses? Explain how you know.

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

**Solution:** Yes they are inverses. We know that is true by multiplying the two matrices and seeing that we get the identity matrix out. To see that:

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} &= \begin{bmatrix} 3 \cdot 2 + 1 \cdot (-5) & 3 \cdot (-1) + 1 \cdot 3 \\ 5 \cdot 2 + 2 \cdot (-5) & 5 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

10. Determine graphically the solution set for each system of inequalities and indicate whether the solution set is bounded or unbounded:

$$\begin{aligned} x - y &< 7 \\ 3x + 2y &\geq 6 \\ x &\geq 0 \end{aligned}$$

**Solution:** The solution is unbounded. To solve it graphically, you need to graph the three lines: $x - y = 7$ (which passes through $(0, -7)$ and $(7, 0)$, graphed with a dashed line), the line $3x + 2y = 6$ (which passes through $(0, 3)$ and $(2, 0)$, and the line $x = 0$ (the $y$-axis). After checking a test point, we see that the solution is the shaded region to the upper right of the triangle. including the two solid lines as boundary. (I will try and get a picture solution on the web soon, but I am having trouble getting one converted to the type of file I need to do so.)

11. True or False (Circle the letter). Partial credit will be given only if an explanation is included.

**T** **F** The matrix $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right]$ corresponds to a system of equations with no solutions.
**Solution:** False, the equations become $x = 2$ and $0 = 0$. Since we don't have a contradiction in the equations, and there are two variables with the second column has no leading 1, there are infinitely many solutions.

**T** **F** The matrix $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}\right]$ corresponds to a system of equations with exactly one solution.
**Solution:** True. The corresponding equations are: $x = 2$ and $y = 0$ and there are exactly 2 variables there is a unique solution. Alternatively, since every column to the left of the vertical line has a leading 1 and there is not a row with corresponding equation $0 = 1$, there must be a unique solution.

**T** **F** The matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 7 \end{array}\right]$ is in row-reduced echelon form.
**Solution:** False. The first non-zero entry in the last row is a 2, not a 1 as required.

**T** **F** The matrix $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ is in row-reduced echelon form.

**Solution:** True. Every row has a leading 1 and the column with the leading 1 has 0s elsewhere.

**T** **F** If $A$ is an $m \times n$ matrix and $B$ is an $m \times k$ matrix, then $AB$ is an $n \times k$ matrix.

**Solution:** False, you cannot multiply $A$ and $B$ since $A$ has a different number of columns as $B$ has rows (unless $m = n$).

12. If $A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$, find the product matrix $AB$.

**Solution:** The product of the matrices is given by

$$\begin{bmatrix} -2 \cdot 1 + 1 \cdot 4 & -2 \cdot (-1) + 1 \cdot (-3) \\ 3 \cdot 1 + 2 \cdot 4 & 3 \cdot (-1) + 2 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 11 & -9 \end{bmatrix}$$

# Calculator and Study Sheet Allowed

13. Write the equation of the line passing through the points $(3, 5)$, and $(-3, 17)$.

**Solution:** The slope of the line is given by

$$m = \frac{17 - 5}{-3 - 3} = \frac{12}{-6} = -2.$$

Using the point-slope equation of the line we obtain

$$(y - 5) = -2(x - 3).$$

14. What is the slope of the line $3x + 5y = 9$?

**Solution:** We could rewrite the line in point slope form, find two points on the line and calculate the slope from those points, or remember the formula for finding the slope from the general equation of a line. In any case, the answer is that the slope is

$$m = \frac{-3}{5}.$$

15. A manufacturer has monthly fixed costs of $2,600$ and a production cost of \$2 for each unit produced. If the product sells for 5 per unit, write

(a) The cost function $C(x)$.

(b) The Revenue Function $R(x)$.

(c) The Profit Function $P(x)$.

(d) What is the break-even point for this system?

(e) The Profit at the break-even point is.

**Solutions:** The fixed cost $F = 2600$ and unit cost 2. Thus the cost function is given by

$$C(x) = 2x + 2600.$$

Since each unit sells for 5 dollars, the revenue function is

$$R(x) = 5x.$$

The Profit function is the difference of the revenue and the cost, so

$$P(x) = R(x) - C(x) = 3x - 2600.$$

The break-even point is where the cost equals the revenue. Thus we solve the equation:

$$5x = 2x + 2600.$$

Solving we get

$$\begin{aligned} 3x &= 2600 && \text{subtracting } 2x \text{ from both sides} \\ x &= \frac{2600}{3} = 866.\overline{6} && \text{dividing by 3.} \end{aligned}$$

Since we are selling units and presumably cannot sell 2/3 of a unit, I would accept either $866\frac{2}{3}$ or 867 units. The profit at this point is then given by

$$P(867) = 3(867) - 2600 = 1.$$

That is the profit at the "break-even" point is 1 dollar if we take 867. More typically we would have that the break even point would have a profit of 0 dollars, which is what happens at $866\frac{2}{3}$.

16. Use any method you like to find the point of intersection of the lines $2x + 5y = -1$ and $3x + y = -8$.

**Solution:** One option is to solve the second equation for $y$, to get

$$y = -3x - 8.$$

Substituting this into the first equation we have

$$\begin{aligned} 2x + 5(-3x - 8) &= -1 \\ 2x - 15x - 40 &= -1 \\ -13x &= -1 + 40 \quad \text{adding 40 to both sides} \\ -13x &= 39 \\ x &= -3. \end{aligned}$$

Substituting $x = -3$ in the first equation we have

$$2(-3) + 5y = -1.$$

Solving this we have $y = 1$. Thus the solution is $x = -3$ and $y = 1$. We can check the answer with the second equation to see that this solution is correct.

A second method would be to use matrices.

$$\begin{aligned} &\left[\begin{array}{cc|c} 2 & 5 & -1 \\ 3 & 1 & -8 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 3 & 1 & -8 \\ 2 & 5 & -1 \end{array}\right] \\ \xrightarrow{R_1 - R_2} &\left[\begin{array}{cc|c} 1 & -4 & -7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -4 & -7 \\ 0 & 13 & 13 \end{array}\right] \\ \xrightarrow{\frac{1}{13}R_2} &\left[\begin{array}{cc|c} 1 & -4 & -7 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 + 4R_2} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \end{aligned}$$

This gives us $x = -3$ and $y = 1$.

Of course, if you have a calculator, you could set this one up with the calculator. However, you might have to solve a $2 \times 2$ on the no calculator/study sheet allowed portion.

The third method would be elimination. To find the $y$ value we would multiply the first equation by 3, the second equation by 2, and then subtract to obtain:

$$13y = 13$$

and thus $y = 1$. We could either substitute in for $y$ or multiply the second equation by 5 and subtract the first equation from it to obtain

$$13x = -39$$

or $x = -3$, and again the solution is $x = -3$ and $y = 1$.

17. Consider the system

$$\begin{aligned} 2x - 4y &= -6 \\ -3x + 6y &= k. \end{aligned}$$

For what values of $k$ does the system have

(a) no solutions?

(b) exactly one solution?

(c) infinitely many solutions?

The easiest way to solve this is to first simplify the first equation by dividing both sides by 2. We have that the equations are

$$\begin{aligned} x - 2y &= -3 \\ -3x + 6y &= k \end{aligned}$$

Note that if we now multiply the first equation by $-3$ we get

$$\begin{aligned} -3x + 6y &= 9 \\ -3x + 6y &= k \end{aligned}$$

The only way that we can have any solutions is if $k = 9$, but in that case there would be infinitely many solutions. Hence:

(a) There are no solutions is $k \neq 9$.

(b) There is no value for $k$ which gives exactly 1 solution.

(c) There are infinitely many solutions if $k = 9$.

18. Write the augmented matrix for the system

$$\begin{aligned} 2x - z &= 1 \\ 3x + 2y &= 4. \end{aligned}$$

Be careful!!

**Solution:**

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 3 & 2 & 0 & 4 \end{array}\right]$$

19. For what values of $k$, $m$, and $n$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ k & m & n & 3 \end{array}\right]$ in row reduced form?

    **Solution:** To have row reduced form, we need to have either $k = 0$, $m = 1$, and $n$ can be anything, or $k = 0$, $m = 0$, and $n = 1$.

    The requirement is that each row should have a leading 1 and that any column with a leading 1 should have 0s elsewhere. This means that $k = 0$ since it is below a leading 1. Since row two must have a leading 1, then $m = 1$ or $m = 0$ and $n = 1$. Now if $m = 1$, then $n$ can be anything since it will not be in a column with a leading 1.

20. Use the indicated row operations to fill in the 5 missing entries in the second matrix below:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 3 & -4 \\ 0 & -4 & -1 & -6 \end{array}\right] \underset{R_3 + 4R_2}{\overset{R_1 - 2R_2}{\longrightarrow}} \left[\begin{array}{ccc|c} 1 & & -5 & \\ 0 & 1 & 3 & -4 \\ 0 & & & \end{array}\right]$$

**Solution:** The matrix should be:

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 15 \\ 0 & 1 & 3 & -4 \\ 0 & 12 & 11 & -22 \end{array}\right]$$

21. **(a)** Write down the augmented matrix for

$$\begin{aligned} x - z &= -2 \\ y + 2z &= 7 \\ -x + 2z &= 6. \end{aligned}$$

**Solution:** The matrix would be

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 7 \\ -1 & 0 & 2 & 6 \end{array}\right]$$

**(b)** Write down what operation should be the first row operation to undertake in Gauss-Jordan elimination.
**Solution:** The first row operation should be $R_3 + R_1$. This would make the entry in the $3, 1$ position a 0.

22. Sandy has a total of \$3,500 on deposit with two savings institutions. One pays interest at the rate of 6% per year and the other pays interest at the rate of 7% per year. If Sandy earned a total of \$250 in interest during a single year, how much does she have on deposit in each institution?
**Solution:** There was an error in this problem. The question should have read that the interest paid was \$220 not \$250. Below I will solve the problem with \$250, and you will see why there must have been an error. Then I will give the solution with 220 instead.

We will let $x$ denote the amount of money in the first institution (paying 6%) and $y$ denote the amount of money in the second institution (paying 7%). Since her total is 3500 we have the equation

$$x + y = 3500.$$

That the interest paid was 250 dollars, gives the equation

$$.06x + .07y = 250.$$

Thus we have the equations

$$\begin{aligned} x + y &= 3500 \\ .06 + .07y &= 250 \end{aligned}$$

This gives us the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 3500 \\ .06 & .07 & 250 \end{array}\right]$$

Row reducing this on our calculator yields the matrix

$$\left[\begin{array}{cc|c} 1 & 0 & -500 \\ 0 & 1 & 4000 \end{array}\right]$$

This matrix corresponds to Sandy investing $-500$ dollars in the account paying 6% interest, which is ludicrous. Hence we know that the problem (or our solution) must have an error.
**Note:** For this problem, it really is that problem that has the error, but if it was a solution error that you didn't have time to fix (i.e., if you see this happen on a test) you should tell me that you know the answer is wrong and why. Checking your answer and discovering it is wrong is worthwhile, and I take off fewer points for an incorrect answer that you can explain why you know it is wrong.

Now for the correct problem, the initial matrix should be

$$\left[\begin{array}{cc|c} 1 & 1 & 3500 \\ .06 & .07 & 220 \end{array}\right]$$

Row reducing this matrix yields

$$\left[\begin{array}{cc|c} 1 & 0 & 2500 \\ 0 & 1 & 1000 \end{array}\right]$$

This corresponds to the equations $x = 2500$ and $y = 1000$. Thus there must be \$2500 invested in the first institution and \$1000 invested in the second.

23. Below is a row-reduced matrix. Write down the system of equations corresponding to this matrix, and then write the solution for the corresponding equations using the parameter $t$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

**Solution:** The equations corresponding to the matrix are

$$\begin{aligned} x + 2y &= -3 \\ z &= 4 \end{aligned}$$

Since there is no leading one in the second column, we will begin by setting

$$y = t.$$

Now solving for the other two equations we have the solutions are

$$\begin{aligned} x &= -3 - 2t \\ y &= t \\ z &= 4. \end{aligned}$$

24. If $2\begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 3y & 2 \\ 0 & -8 \end{bmatrix}$, find $x$ and $y$.
**Solution:** Using the matrix operations, we have

$$\begin{bmatrix} 2x & 2 \\ 0 & 2y \end{bmatrix} = \begin{bmatrix} 3y & 2 \\ 0 & -8 \end{bmatrix}.$$

Setting the terms equal to each other, we have

$$\begin{aligned} 2x &= 3y \\ 2y &= -8 \end{aligned}$$

The second equation implies that $y = -4$. Substituting into the first equation we have $x = -6$.

25. Write down the corresponding matrix equation to the system of linear equations below, and solve it (via calculator) using the inverse matrix (show the inverse matrix):

$$\begin{aligned} 2x + 3y + 4z &= 12 \\ x - 5y + 2z &= 4 \\ 3x - y + z &= 1 \end{aligned}$$

**Solution:** In matrix form, this is

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 2 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix}.$$

To solve, we enter the matrix (call it $A$) in our calculator to obtain

$$A^{-1} = \begin{bmatrix} \frac{-3}{65} & \frac{-7}{65} & \frac{2}{5} \\ \frac{1}{13} & \frac{-2}{13} & 0 \\ \frac{14}{65} & \frac{11}{65} & \frac{-1}{5} \end{bmatrix}.$$

Multiplying the solution column vector by this matrix gives solution:

$$\begin{aligned} x &= \frac{-38}{65} \\ y &= \frac{4}{13} \\ z &= \frac{199}{65}. \end{aligned}$$

26. A T-shirt company wants to manufacture 2 types of T-shirts. The first T-shirt requires 10 minutes on machine $A$, 5 minutes on machine $B$, and 3 minutes on machine $C$. The second T-shirt requires 7 minutes of manufacturing time on machine $A$, 6 minutes of manufacturing time on machine $B$, and 4 minutes of manufacturing time on machine $C$. The first T-shirt sells for a profit of 5 dollars and the second T-shirt sells for a profit of 6 dollars. Set up the linear programming problem for this company. Label all of your variables.

    **Correction to question:** Oops, I didn't include the amount of time on each machine... sorry. Suppose machine A has 2 hours available, machine B has 1 hour available, and machine C has 3 hours available.

    **Solution:** Let $x$ denote the number of the first T-shirt to be made, and $y$ denote the number of the second T-shirt to be made. For the requirements on machine A, we know

$$10x + 7y$$

For machine B we have $5x + 6y$ and for machine C, we have $3x + 4y$. Of course since there aren't any negative numbers of T-shirts, $x \geq 0$ and $y \geq 0$. We are trying to maximize the profit function $P = 5x + 6y$. Thus the final set of equations is:

$$\begin{aligned} P &= 5x + 6y \\ 10x + 7y &\leq 120 \\ 5x + 6y &\leq 60 \\ 3x + 4y &\leq 180. \end{aligned}$$