Knot Mosaics: Results and Open Questions

Lew Ludwig
Denison University

Our program

- Review of knot mosaics
- Some recent results in knot mosaics
  - Arc presentation
  - Grid diagram
- Provide an upper bound for the mosaic number of an infinite family of knots
- Determine the mosaic number for an infinite family of knots
- Conclude with open questions
What are Knot Mosaics?

- Lomonaco and Kauffman (2008)

- Kuriya (2008), Shebab (2012) tame knot theory and mosaic knot theory are equivalent

Some terminology

- **mosaic number** of a knot $K$, denoted $m(K)$

The minimal size mosaic board that a knot will fit on

$m(3_1)=4 \quad m(4_1)=5$
Question – Lomonaco and Kauffman

- Is the mosaic number, $m(K)$, related to the crossing number, $c(K)$, of a knot $K$?

Recent results:
(H.J. Lee, K. Hong, H. Lee, and S. Oh)

Let $K$ be a nontrivial knot (or a non-split link except the Hopf link and the $6_3^3$), then

$$m(K) \leq c(K) + 1$$

If $K$ is prime and non-alternating, then

$$m(K) \leq c(K) - 1$$
Some useful tools

(Bae-Park, 2000) Let $K$ be a knot or a non-split link, then

$$\alpha(K) \leq c(K) + 2$$

If $K$ is prime and non-alternating, then

$$\alpha(K) \leq c(K) + 1$$

(Jin-Park, 2010) Let $K$ be a non-alternating prime knot or link, then

$$\alpha(K) \leq c(K)$$

What is $\alpha(K)$?

- Arc presentation
  - $z$-axis is binding
  - pages are half-planes
  - finitely many pages
  - each page meets one arc

Arc index, $\alpha(K)$, minimum number of pages required
Some useful tools

(Bae-Park, 2000) Let $K$ be a knot or a non-split link, then

$$\alpha(K) \leq c(K) + 2$$

If $K$ is prime and non-alternating, then

$$\alpha(K) \leq c(K) + 1$$

(Jin-Park, 2010) Let $K$ be a non-alternating prime knot or link, then

$$\alpha(K) \leq c(K)$$

One more tool – grid diagrams

- Grid diagrams are $n \times n$
- One $X$ in every row (column)
- One $O$ in every row (column)
- Vertical lines are overcrossings
- Grid Index, $G(K)$, min # vertical segments

★Natural connection to knot mosaics★
Cromwell grid moves (Dynnikov)

- Stabilization and destabilization
- Interchanging neighboring edges if their pairs of endpoints do not interleave
- ★Cyclic permutation of vertical (horizontal) edges – do not change $G(K)$

Keep goal in mind...

Knot $K$: $m(K) \leq c(K) + 1$

Knot $K$, prime non-alternating: $m(K) \leq c(K) - 1$

One more connection: grid diagram and arc presentation

★Arc index $(K) = \text{Grid Index } (K)$★
Now for the proof: $m(K) \leq c(K) + 1$

Notice the horizontal arcs:

Fig. 1

Fig. 2

Now for the proof: $m(K) \leq c(K) + 1$

Notice the horizontal arcs Fig. 2:

Reduced the mosaic size by 1
Now for the proof: $m(K) \leq c(K) + 1$

Notice the horizontal arcs Fig.1:

Fig. 1
Now for the proof: $m(K) \leq c(K) + 1$

Notice the horizontal arcs Fig.1:

![Figure 1](image1)

In either case...

So $m(K) = \alpha(K) - 1$

$\leq (c(K) + 2) - 1$

$= c(K) + 1$

and $m(K) \leq c(K) - 1$ if non-alt prime

Bae & Park

Jin & Park
A bound on the mosaic number of an infinite family of knots

- (L. & Wu, 2012)
  \[ m(T_{(p,p+1)}) \leq 2p \]

- (H.J. Lee, K. Hong, H. Lee, and S. Oh)
  \[ m(T_{(p,q)}) \leq p+q-2 \quad \text{if } |p-q| \neq 1 \]

\[ T_{(5,3)} \]

The mosaic number of an infinite family of knots

- Is there an infinite family of knots whose mosaic number is realized only when the crossing number is not?

- Why is this interesting?
  - **Unknotting number** — minimum number of times knot must pass through itself to unknot
  - Bernhard 1994, generalized

  Nakanishi 1983 result — infinite family of knots whose unknotting number is realized when the crossing number is NOT!
Our Construction

- Number of Crossings: 6
  Mosaic Size: 6

- Number of Crossings: 7
  Mosaic Size: 5

- Number of Crossings: 22
  Mosaic Size: 8

- Number of Crossings: 23
  Mosaic Size: 7

What is our Game Plan?

- $L_7$

- $L_9$

- $L_{11}$

- $L_{13}$
Claim: $L_{2n+1}$ is the family we seek, Three Acts

1. Must compute crossing number for this family.
2. Must compute mosaic number for this family.
3. Must show when mosaic number is realized, crossing number is not.

Act 1: Crossing Number

- $(2n-1)^2$ inner tiles
- $(2n-1)^2 - 2$ crossing tiles
- Make reduced alternating, remove one crossing
- $c(L_{2n+1}) = (2n-1)^2 - 3$
- $(c(L_7) = 22)$
Act 2: Mosaic Number

- Claim: $m(L_{2n+1}) = 2n+1$
- By Act 1, need $(2n-1)^2 - 3$ crossings
- A $2n$-mosaic board has $(2n-2)^2$ possible crossings
- Since $(2n-2)^2 < (2n-1)^2 - 3$
- $m(L_{2n+1}) = 2n+1$

Act 3

- We must show that when the mosaic number is realized, the crossing number is not.
- Important fact: $L_{2n+1}$ is a reduced, alternating knot
Why are reduced, alternating Knots a big deal?

**Tait Flyping Conjecture:**
Given any two reduced alternating diagrams $D_1$ and $D_2$ of an oriented, prime alternating knot, $D_1$ may be transformed to $D_2$ by a sequence of flypes.

(Thistlethwaite & Menasco 1991)

---

Why are reduced, alternating Knots a big deal?

- Tait Flyping Conjecture
- $L_7$ has 1 possible flype
Conclusion…

- There are only two “versions” of $L_{2n+1}$

- How can the individual versions be placed on a mosaic board and maintain their crossings?

$m$-gons are preserved on sphere

![Diagram of trefoils on a sphere]

Trefoil  Trefoil
What about m-gons on $L_{2n+1}$?

Lucky break 1:
Only 2-, 3-, 4-, 5-, and (8n-11)-gons AND (8n-11) m-gon must be on outside.

Ex: 2-gon won’t work

... and another lucky break.

Lucky break 2:
Both reduced alternating projections have a 5-gon.
Back to Act 3

- We know a non-reduced, non-alternating $L_7$ can fit on a 7x7 board, what about a reduced, alternating version of $L_7$?

Can a reduced, alternating $L_7$ fit on a 7x7?

- We need to place 22 crossing tiles
- Plus three non-crossing tiles as inner tiles
- Since the $8n-11=13$-gon must be on the outside of the knot, and there are 16 inside perimeter tiles, all 3 inner non-crossing tiles must be along inside perimeter

Where do the 3 non-crossing tiles go?
How can we place the three non-crossing tiles?

- \( \binom{7}{2} = 21 \) ways to place other two non-crossing tiles
- Breaks into 6 cases
- Ex: both on a corner, suitably connected, no 5-gon
- Other 5 cases are similar, either no 5-gon or not suitably connected

The close of Act 3…

- We found an infinite family of knots whose mosaic number is only realized when the crossing number is not.

\[ L_{2n+1} \]
Why not even mosaic boards?

- $L_{12}$ – 10 component link
- $L_{12}$ – 9 component link

New family of knots (L. & H.J. Lee)

- $H_{2n}$- helix knots
  - Alternating knot
  - $m(H_{2n})=2n$
  - $c(H_{2n})=4n^2-10n+7$

$n=6$
Family of knot from $H_{2n}$ (L. & H.J. Lee)

1. $HL_{2n}$
   - Alternating knot, but non-alternating diagram
   - $m(\text{HL}_{2n}) = 2n$
   - $c(\text{HL}_{2n}) = 4n^2 - 10n + 6$
   - Mosaic number realized, crossing number not

2. $Haa_{2n}$
   - Almost alternating knot
   - $m(L_{2n}) = 2n$
   - $c(L_{2n}) = 4n^2 - 10n + 6$
   - Mosaic number realized, crossing number not

$n = 6$
Family of knot from $H_{2n}$ (L. & H.J. Lee)

- $Haa_{2n}$
  - Almost alternating knot
  - $m(L_{2n}) = 2n$
  - $c(L_{2n}) = 4n^2 - 10n + 6$
  - Mosaic number realized, crossing number not

$n=6$
Open Questions

- What is the mosaic number for (2,q)-torus knots?
- (p,p+1)-torus knots?
- Can the crossing number be used for determining the mosaic number?
- Does there exist a knot whose mosaic number is $n$, but whose crossing number is only realized on a mosaic board of size $n+2$?

Acknowledgements

- Erica, Blake, and Ramin
- Hwa Jeong Lee – KAIST
- Erica Evans, Joe Paat, Jacob Shapiro, Gary Wu
- Colin Adams – Williams College
- Denison University Anderson Endowment
- Rich Ligo – University of Iowa

Thanks!
References