Exactly 14 intrinsically knotted graphs have 21 edges

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Definitions and Related Results I

We will consider a graph as an embedded graph in $\mathbb{R}^3$.

- A graph $G$ is called \textit{intrinsically knotted (IK)} if every embedding of the graph contains a knotted cycle.

Conway-Gordon’85

$K_7$ is IK
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Definitions and Related Results II

- $\nabla Y$ move

Diagram:

- Points labeled $a$, $b$, and $c$ form a triangle.
Definitions and Related Results II

- $\nabla Y$ move

\[ a \rightarrow b \rightarrow c \rightarrow a \]
• $\nabla Y$ move
Definitions and Related Results II

- \( \nabla Y \) move

\[ a \quad c \quad b \quad v \]

Motwani-Raghunathan-Saran’88

\( \nabla Y \) move preserves IKness.

We will only consider triangle-free graphs in this work.
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We will only consider triangle-free graphs in this work.
A graph $H$ is a *minor* of another graph $G$ if it can be obtained from $G$ by deleting edges and vertices and by contracting edges.

Provided that a graph $G$ is IK and has no proper minor which is IK, $G$ is said to be *minor minimal intrinsically knotted* (MMIK).

$K_7$ and the thirteen graphs obtained from it by $\nabla Y$ moves are MMIK.
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Motivations

Johnson-Kidwell-Michael’07

Any IK graphs consists at least 21 edges.

- It is sufficient to consider simple and connected graphs having no vertex of degree 1 or 2.

Hanaka-Nikkuni-Taniyama-Yamazaki, Goldberg-Mattman-Naimi

They constructed twenty graphs derived from $H_{12}$ and $C_{14}$ by $Y\bar{Y}$ moves and showed that these six graphs $N_9, N_{10}', N_{10}'', N_{11}, N_{11}', N_{12}$ and $N_{11}'$ are not IK.
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Main Results

**Main Theorem I**
The only triangle-free IK graphs with 21 edges are $H_{12}$ and $C_{14}$.

**Main Theorem II**
Only $K_7$ and the thirteen graphs obtained from $K_7$ by $\nabla Y$ moves are IK graphs with 21 edges.
Terminology

Let $G = (V, E)$ be a triangle-free graph with 21 edges.

For any two distinct vertices $a$ and $b$,

- $\text{deg}(a)$: the degree of a vertex $a$.
- $\text{dist}(a, b)$: the number of edges in the shortest path connecting them.

$\text{dist}(a, b) = 2$
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A graph is **2-apex** if one can remove 2 vertices from it to obtain a planar graph.

If $G$ is a 2-apex, then $G$ is not IK.

$\hat{G}_{ab} = (\hat{V}_{ab}, \hat{E}_{ab})$ : the graph obtained from $G$ by deleting two vertices $a$ and $b$, and then contracting one edge incident to a vertex of degree 1 or 2 repeatedly until no vertices of degree 1 or 2 exist.
Terminology

A graph is 2-apex if one can remove 2 vertices from it to obtain a planar graph.

Blain-Bowlin-Fleming-Hendricks-Lacombe’07, Ozawa-Tsutsumi’07

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If $G$ is a 2–apex, then $G$ is not IK.

- $\tilde{G}_{a,b} = (\tilde{V}_{a,b}, \tilde{E}_{a,b})$: the graph obtained from $G$ by deleting two vertices $a$ and $b$, and then contracting one edge incident to a vertex of degree 1 or 2 repeatedly until no vertices of degree 1 or 2 exist.
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To count the number of edges of $\hat{G}_{a,b}$, we have some notations.

- $E(a)$: the set of edges which are incident to $a$.
  \[|E(a)| = 6\]

- $V(a) = \{ c \in V \mid \text{dist}(a, c) = 1\}$
  \[V(a) = \{ c_1, c_2, c_3, c_4, c_5, c_6 \}\]

- $V_n(a) = \{ c \in V \mid c \in V(a), \deg(c) = n\}$
  \[V_3(a) = \{ c_5, c_6 \}\]

- $V_n(a, b) = V_n(a) \cap V_n(b)$
  \[V_3(a, b) = V_3(a) \cap V_3(b) = \{ c_5, c_6 \}\]
  \[V_4(a, b) = V_4(a) \cap V_4(b) = \{ c_4 \}\]

- $V_Y(a, b) = \{ c \in V \mid \exists d \in V_3(a, b) \text{ s.t. } V_3(d) = \{ a, b, c \}\}$
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\[ |\widehat{E}_{a,b}| \]
\[ |E_{a,b}| \leq 21 - |E(a) \cup E(b)| \]
\[ |E_{a,b}| \leq 21 - |E(a) \cup E(b)| - (|V_3(a)| + |V_3(b)| - |V_3(a, b)|) - |V_4(a, b)| \]
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\[ |E_{a,b}| = 21 - |E(a) \cup E(b)| - (|V_3(a)| + |V_3(b)| - |V_3(a, b)|) + |V_4(a, b)| - |V_Y(a, b)| \]
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A count equation in $\tilde{G}_{a,b}$

$$|\tilde{E}_{a,b}| = 21 - |E(a) \cup E(b)| - (|V_3(a)| + |V_3(b)| - |V_3(a, b)| + |V_4(a, b)| + |V_Y(a, b)|)$$

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A graph is \( n\)-apex if one can remove \( n \) vertices from it to obtain a planar graph.

\[\text{Blain-Bowlin-Fleming-Hendricks-Lacombe’07, Ozawa-Tsutsumi’07}\]

If \( G \) is a 2–apex, then \( G \) is not IK.

**Lemma 1.**
If \(|\hat{E}_{a,b}| \leq 8\), then \( \hat{G}_{a,b} \) is a planar graph. Thus \( G \) is not IK.

**Lemma 2.**
If \(|\hat{E}_{a,b}| = 9\), then \( \hat{G}_{a,b} \) is either a planar graph or homeomorphic to \( K(3, 3) \). Furthermore if \( \hat{G}_{a,b} \) is not homeomorphic to \( K(3, 3) \), then \( G \) is not IK.

Note: \( K(3, 3) \) is a triangle-free graph and every vertex has degree 3.
Useful Lemmas

A graph is $n$-apex if one can remove $n$ vertices from it to obtain a planar graph.

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If $|\hat{E}_{a,b}| = 9$, then $\hat{G}_{a,b}$ is either a planar graph or homeomorphic to $K(3, 3)$. Furthermore if $\hat{G}_{a,b}$ is not homeomorphic to $K(3, 3)$, then $G$ is not IK.

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Note: $K(3, 3)$ is a triangle-free graph and every vertex has degree 3.
Our Process

To prove Theorem 1, we will show that any triangle-free graph with 21 edges is eventually either a 2-apex or homeomorphic to one of $H_{12}$ and $C_{14}$.

1. Constructing all possible such triangle-free graph $G$ with 21 edges,
2. Deleting two suitable vertices $a$ and $b$ of $G$,
3. Counting the number of edges of $\hat{G}_{a,b}$.

- If $\hat{G}_{a,b}$ has 9 edges or less, we could use Lemma 1. or Lemma 2. in order to show that $G$ is not IK.
- In the event that $\hat{G}_{a,b}$ is not planar, we will show that $G$ is homeomorphic to $H_{12}$ or $C_{14}$. 
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Sketch of Proof of Theorem 1.

Throughout this proof, \( a \) denotes one of vertices with maximal degree in \( G \).

The proof is divided into three parts according to the degree of \( a \).

I. Any graph \( G \) with \( \text{deg}(a) \geq 5 \) cannot be IK.

II. Only IK graph with \( \text{deg}(a) = 4 \) is \( H_{12} \).

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We will show that for some $a, b \in V$ either $|\hat{E}_{a,b}| \leq 8$ or $|\hat{E}_{a,b}| = 9$ but $\hat{G}_{a,b}$ is not homeomorphic to $K(3, 3)$.

**Lemma 1.**

If $|\hat{E}_{a,b}| \leq 8$, then $\hat{G}_{a,b}$ is a planar graph. Thus $G$ is not IK.

**Lemma 2.**

If $|\hat{E}_{a,b}| = 9$, then $\hat{G}_{a,b}$ is either a planar graph or homeomorphic to $K(3, 3)$. Furthermore if $\hat{G}_{a,b}$ is not homeomorphic to $K(3, 3)$, then $G$ is not IK.

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I. \( \text{deg}(a) \geq 5 \)

I.1. \( \text{deg}(a) \geq 6 \) or \( \text{deg}(a) = 5 \) with \( |V_3(a)| \geq 4 \)

I.2. \( \text{deg}(a) = 5 \) and \( |V_3(a)| = 3 \)

I.3. \( \text{deg}(a) = 5 \) and \( |V_3(a)| = 0 \)

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\[
|\widehat{E}_{a,b}| = 21 - |E(a) \cup E(b)| - (|V_3(a)| + |V_3(b)| - |V_3(a, b)| + |V_4(a, b)| + |V_Y(a, b)|)
\]

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If \( \deg(a) \geq 6 \), then \( |V_3(a) \leq 3| \). Let \( c \) be any vertex in \( V_3(a) \).

- \( |\widehat{E}_{a,b}| \leq 21 - 9 \)
- \( |\widehat{E}_{a,b}| \leq 21 - 9 - (3) \)
- \( |\widehat{E}_{a,b}| \leq 21 - 9 - (3 + 1) = 8 \)
- \( |\widehat{E}_{a,b}| \leq 21 - 10 \)
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- $|E_{a,c}| \leq 21 - 9 - (3)$
- $|E_{a,d}| \leq 21 - 9 - (3 + 1) = 8$
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![Diagram showing the change in graph structure](https://via.placeholder.com/150)

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II. $\text{deg}(\alpha) = 4$

Let $V_n$ denote the set of vertices of degree $n$.

Since $|V| = |V_4| + |V_3|$ and $4|V_4| + 3|V_3| = 2|E|$, the pair $(|V_4|, |V_3|)$ has three choices $(3, 10), (6, 6)$ and $(9, 2)$.

We show that $G$ of type $(|V_4|, |V_3|) = (3, 10)$ or $(|V_4|, |V_3|) = (9, 2)$ is not IK and $G$ possibly is $H_{12}$ when $(|V_4|, |V_3|) = (6, 6)$.

II.1. $(|V_4|, |V_3|) = (3, 10)$

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\end{enumerate}
III. $\text{deg}(a) = 3$

Since $|E| = 21$ and every vertex has degree 3, there are exactly 14 vertices.

Lemma 3.
The distance between any pair of vertices cannot exceed 3.

* $V(a) = \{b_1, b_2, b_3\}$

* $V(b_i) = \{a, c_{2n-i}, c_{2i}\}$

* $V \setminus (V(a) \cup V(b_i)) = \{d_1, d_2, d_3, d_4\}$
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**Lemma 3.**
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This graph is exactly $C_{14}$. 
Thank you for listening!