Numerical Analysis and Simulation of Resource-Exploration Models

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Abstract

In this paper, we examine models for exploration and consumption of resources. The fundamental feature of the models is the jump-process nature of the exploration for and discovery of the resource. Several models have been proposed and analyzed in the literature. Here we provide numerical schemes, convergence properties, and some new models that provide risk-averse policies to avoid depletion of the resource.

1 Introduction

The use of non-renewable resources presents a variety of political and economic problems to planners. In this paper, we examine some models for the optimal exploration for and consumption of non-renewable natural resources. Relying on fairly general basic well-posedness results found, e.g., in [5] and [16], we consider some models representing risk-averse optimization, and we develop numerical approximation and convergence results for these optimzation problems.

Non-renewable resource management is a major challenge in today's world. Energy demands, in particular, continue to expand, stressing the oil and natural gas markets. Fundamental efforts in modeling and analysis are contained

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in [5, 16] and the references therein. In [6], analysis via piecewise deterministic processes is conducted in a general setting. In [8], a deterministic problem of extraction from multiple reserves with different extraction costs is examined. Recent efforts also include [10, 14], in which optimal exploration for one firm in the presence of taxation and competition is studied. The analysis of [9] equates resource extraction to a call option.

The dynamics of the processes we consider contain two basic components, discovery and consumption. Consumption is modeled as a continuously varying quantity, in terms of a (controlled) rate of consumption. Discovery, on the other hand, is a jump process, whose jump rates and jump levels depend on the (controlled) exploration effort. The dynamic equation is of the form

$$dX(t) = dI_t - c_t \, dt, \tag{1}$$

or

$$X(t) = X_0 + I_t - \int_0^t c_\tau \, d\tau, \tag{2}$$

in which X denotes the stock of the resource, I denotes the cumulative amount discovered through exploration, and c denotes the rate of consumption of the resource.

The control policies to be determined are the consumption rate c, which we assume to be constrained to lie in a bounded interval $[0, \bar{c}]$, and the exploration effort, e. The manner in which this control variable enters the equation involves the nature of the jump discovery process, I. We assume here that the rate of discovery is proportional to the effort, and that the amount of discovery is independent of the effort. For simplicity of exposition, we assume that the discovery amount is an absolutely continuous random variable with density $q\colon (0,\infty) \to [0,\infty)$, assumed to have finite mean μ_q and variance σ_q^2. Thus, the jump process is modeled as

$$Pr[I(t+h) = y | I(t) = y, e] = 1 - \lambda e h + o(h) \tag{3}$$

$$Pr[I(t+h) \in (y+a, y+b] | I(t) = y, e] = \lambda e h \int_a^b q(z) \, dz + o(h) \tag{4}$$

in which λ denotes the rate constant for discovery, over the small time period $[t, t+h]$. Such a process can be defined rigorously in terms of its jump times.

We assume that $T_1, T_2, T_3, \cdots,$ is a sequence of positive random variables satisfying the following:

$$Pr[T_1 \le t] = 1 - \exp\{-\int_0^t \lambda e(s)\, ds\} \quad (5)$$

$$Pr[T_{n+1} - T_n \le t | T_1, \cdots, T_n] = 1 - \exp\{-\int_0^t \lambda e(T_n + s)\, ds\}. \quad (6)$$

For a detailed description of this model (and some of its generalizations), see [5] and [16].

The optimization problem for the process is to determine controls $c\colon [0,\infty) \to [0,\bar{c}]$ and $e\colon [0,\infty) \to [0,\bar{e}]$ in such a way that the functional

$$J(x_0, c, e) = E\Big[\int_0^\infty e^{-\beta t}(U(c(t)) - He(t))\, dt\Big], \quad (7)$$

is maximized. Here β is the usual discount rate, U denotes the utility of consumption, and He denotes the cost of exploration at rate e (assumed linear). Typically U is chosen to be of the HARA (hyperbolic absolute risk averse, see, e.g., [1, 4]) form $U(c) = c^\gamma$, for some $\gamma \in (0,1)$. These functions have infinite derivative at $c = 0$, modeling the notion that pushing consumption rates to 0 involves steeper changes in utility. Generally speaking, when a smaller value of γ is specified, the controls will work harder to keep the system away from 0.

The optimal controls are characterized in a feedback manner, using the value function $V(x_0) = \sup_{c,e} J(x_0, c, e)$, which satisfies the Bellman equation

$$\beta V(x) = \sup_{0\le c\le \bar{c}} \Big\{(U(c) - cV'(x)\Big\} + \sup_{0\le e\le \bar{e}} \Big\{\lambda e \int_0^\infty (V(x+y) - V(x))q(y)\, dy - He\Big\}, \quad (8)$$

whose properties are detailed, e.g., in [16]. In particular, in [16] it is seen that this equation has a unique viscosity solution.

Another interesting property of this model (again, see [16]) is the reachability of 0 resource.

Having recalled the model and its mathematical foundations, we now turn to the problem of developing a numerical scheme and analyzing its convergence properties.

2 A Numerical Scheme Based on Markov Chain Approximation

We begin this section by defining the transition probabilities of a controlled Markow chain, which will approximate the controlled resource process described above. The basic idea is to construct the process so that its moments approximate those of the continuous process. The very general weak convergence theory of Kushner (see, e.g., [13, 11, 12]) can then be applied to obtain convergence of the processes. Here we go directly to the solutions of the discrete Bellman equations. Relying on special structure of the problem, we can obtain convergence results for the feedback control forms as well as the value function.

Given a discrete time step Δt and spatial step h, we define the transition probabilities in the following way. On the discrete grid $0 \leq i, j < \infty$, we let $P_{i,j}^{c,e}$ denote the one step transition probability from state i to state j using controls c and e. The, for $i \geq 1$, we set

$$\begin{aligned} P_{i,i-1}^{c,e} &= \frac{c\Delta t}{h}, \\ P_{i,i+j}^{c,e} &= \lambda e \Delta t h q(jh), \quad j \geq 1, \\ P_{i,i}^{c,e} &= 1 - P_{i,i-1}^{c,e} - \sum_{j=1}^{\infty} P_{i,i+j}^{c,e}, \end{aligned}$$

in which $\Delta t = h^2/Q$, with the denominator Q chosen to ensure all quantities are between 0 and 1. In particular, we choose Q as

$$Q = \bar{c}h + \lambda \bar{e} h^3 \sum_{j=1}^{\infty} q(jh).$$

Note that this choice of Q forces $\Delta t \to 0$ as $h \to 0$. At the left boundary, we assume that the state $i = 0$ is absorbing: $P_{0,0} = 1$.

This Markov chain is controlled via the discrete cost functional

$$J^h(x_0, c, e) = E\Big[\sum_{k=1}^{\infty} e^{-\beta \Delta t k}\Big(U(c(t_k)) - He(t_k)\Big)\Delta t\Big], \tag{9}$$

leading to the Bellman equation

$$V_i^h = \sup_{0 \leq c \leq \bar{c}, 0 \leq e \leq \bar{e}} \Big\{ e^{-\beta \Delta t} \sum_j P_{i,j}^{c,e} V_j^h + \Delta t \Big(U(c) - He\Big)\Big\} \tag{10}$$

for the discrete value function. We define the nonlinear functional T^h by

$$T^h(V)_i = \sup_{0\le c\le \bar{c}, 0\le e\le \bar{e}} \Big\{ e^{-\beta\Delta t} \sum_j P^{c,e}_{i,j} V_j + \Delta t \Big(U(c) - He \Big) \Big\} \tag{11}$$

As in [2], we note that T^h defines a contraction on $\mathcal{V} = \{v \in \ell^\infty \colon v_i \ge 0\}$ equipped with the usual sup-norm. We appeal to the contraction mapping principle to obtain a solution for the discrete Bellman equation. Rearranging terms and using the definitions of the transition probabilities, we see that

$$\begin{aligned} V_i^h &= e^{-\beta\Delta t} V_i^h + \Delta t \sup_{0\le c\le \bar{c}} \Big\{ U(c) - e^{-\beta\Delta t} c \frac{V_i^h - V_{i-1}^h}{h} \Big\} \\ &+ \Delta t \sup_{0\le e\le \bar{e}} \Big\{ e \Big(\lambda e^{-\beta\Delta t} \sum_{j=1}^{\infty} (V_{i+j}^h - V_i^h) h q(jh) - H \Big) \Big\} \end{aligned}$$

Bringing the $e^{-\beta\Delta t} V_i^h$ term to the left hand side, and dividing by Δt, we have

$$\begin{aligned} \left(\frac{1 - e^{-\beta\Delta t}}{\Delta t} \right) V_i^h &= \sup_{0\le c\le \bar{c}} \Big\{ U(c) - e^{-\beta\Delta t} c \frac{V_i^h - V_{i-1}^h}{h} \Big\} \\ &+ \sup_{0\le e\le \bar{e}} \Big\{ \lambda e \Big(e^{-\beta\Delta t} \sum_{j=1}^{\infty} (V_{i+j}^h - V_i^h) h q(jh) - h(e) \Big) \Big\} \end{aligned}$$

an equation strongly resembling the continuous Bellman equation. We should remark here that, while this finite difference scheme may appear to be a more natural numerical approximation to the Bellman equation of the continuous problem, the discrete Bellman equation (9) has a control interpretation that allows for value or policy iteration numerical solutions. Moreover, its control interpretation, with the contraction mapping principle, provides the theoretical foundation for solution.

In the following sections, we will establish some properties of the discrete value function and prove that it converges to the value function for the continuous problem. We suggest that the reader who is not interested in technical details skip to 5, in which we describe the numerical implementation and provide computational results.

3 Important Properties of the Discrete Value Function

Our first steps toward proving convergence involve establishing some important properties of the discrete value functions.

Property 1. The value function V^h is a nondecreasing function of i.

PROOF. Our first step is to consider applying T^h to a nondecreasing function $v \in \mathcal{V}$. Let c^* and e^* be the controls optimal for state i, and denote by $P^*_{i,j}$ the transition probablilties associated with those controls. Using the definition of T^h, we have that

$$T^h(v)_{i+1} - T^h(v)_i \quad \geq \quad \Big\{e^{-\beta\Delta t} \sum_{j=i-1}^{\infty} (P^*_{i+1,j+1} - P^*_{i,j})v_j\Big\}.$$

Note that $P^*_{i+1,i+1+j} = P^*_{i,i+j}$ for $i-1 \leq j$. Thus we have

$$T^h(v)_{i+1} - T^h(v)_i \quad \geq \quad e^{-\beta\Delta t} \sum_{j=i}^{\infty} P^*_{i+1,j}(v_j - v_{j-1}).$$

Property 2. The optimal exploration rate e^* is zero for sufficiently large i.

PROOF. Since V^h_i is a bounded and nondecreasing function of i, it must have a limit, V^h_∞, as $i \to \infty$. Coupling this observation with the integrability of the discovery density q, we may choose N in such a way that $\lambda e^{-\beta\Delta t} \sum_{j=N}^{\infty} hq(jh)(V^h_j - V^h_N) < H/2$. The exploration rate is determined by the term

$$\sup_{0\leq e\leq \bar{e}} \Big\{e\Big(\lambda e^{-\beta\Delta t} \sum_{j=1}^{\infty} (V^h_{i+j} - V^h_i)hq(jh) - H\Big)\Big\}$$

For $i \geq N$, then, the quantity multiplying e is less that $-H/2$. Thus, the maximizing value of e in the Bellman equation must be 0 for all $i \geq N$.

Property 3. If U is $C[0,\infty) \cap C^1(0,\infty)$, increasing and concave, with $U(0) = 0$, then the value function V^h is a concave function of i.

PROOF. Using Property 2, we see that the value function satisfies

$$\left(\frac{1 - e^{-\beta\Delta t}}{\Delta t}\right)V^h_i \quad = \quad \sup_{0\leq c\leq \bar{c}} \Big\{U(c) - e^{-\beta\Delta t} c \frac{V^h_i - V^h_{i-1}}{h}\Big\}$$

for sufficiently large i. Define

$$F(p) = \sup_{0 \le c \le \bar{c}} \Big\{ U(c) - e^{-\beta \Delta t} c p \Big\}.$$

Note that F is a nonincreasing function of p, since $U(c) - e^{-\beta\Delta t} cp$ is, for every value of c. Also, note that

$$\begin{aligned}
\left(\frac{1 - e^{-\beta\Delta t}}{\Delta t}\right) V_i^h &= F\Big((V_i^h - V_{i-1}^h)/h\Big) \\
\left(\frac{1 - e^{-\beta\Delta t}}{\Delta t}\right) V_{i+1}^h &= F\Big((V_{i+1}^h - V_i^h)/h\Big)
\end{aligned}$$

leading to

$$\left(\frac{1 - e^{-\beta\Delta t}}{\Delta t}\right)(V_{i+1}^h - V_i^h) = F\Big((V_{i+1}^h - V_i^h)/h\Big) - F\Big((V_i^h - V_{i-1}^h)/h\Big)$$

Since V is nondecreasing, and since F is nonincreasing, we must have

$$(V_i^h - V_{i-1}^h)/h \ge (V_{i+1}^h - V_i^h)/h$$

so that V^h is concave, for sufficiently large i. To obtain the result for all possible i, we proceed with a backward induction argmument (as in [7]). Suppose that

$$(V_i^h - V_{i-1}^h)/h \ge (V_{i+1}^h - V_i^h)/h$$

for all $i \ge n+1$. Our goal is to show that

$$(V_n^h - V_{n-1}^h)/h \ge (V_{n+1}^h - V_n^h)/h.$$

Let e^* be the optimal exploration rate for $n+1$. Then we have

$$\begin{aligned}
\left(\frac{1 - e^{-\beta\Delta t}}{\Delta t}\right) V_{n+1}^h &= e^*\Big(\lambda e^{-\beta\Delta t} \sum_{j=1}^{\infty} (V_{n+1+j}^h - V_{n+1}^h) h q(jh) - H\Big) \\
&+ F\Big((V_{n+1}^h - V_n^h)/h\Big),
\end{aligned}$$

and

$$\left(\frac{1-e^{-\beta\Delta t}}{\Delta t}\right)V_n^h \geq e^*\Big(\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(V_{n+j}^h - V_n^h)hq(jh) - H\Big) + F\Big((V_n^h - V_{n-1}^h)/h\Big),$$

so that

$$\left(\frac{1-e^{-\beta\Delta t}}{\Delta t}\right)(V_{n+1}^h - V_n^h) \leq F\Big((V_{n+1}^h - V_n^h)/h\Big) - F\Big((V_n^h - V_{n-1}^h)/h\Big) + e^*\Big(\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(V_{n+1+j}^h - V_{n+1}^h) - (V_{n+j}^h - V_n^h)hq(jh)\Big)$$

Now, since

$$(V_i^h - V_{i-1}^h)/h \geq (V_{i+1}^h - V_i^h)/h$$

for all $i \geq n+1$, we see by induction that

$$(V_{i+j}^h - V_{i-1}^h)/h \geq (V_{i+1+j}^h - V_i^h)/h$$

for all $i \geq n+1$ and all $j \geq 0$. Thus, we have that

$$e^*\Big(\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(V_{n+1+j}^h - V_{n+1}^h) - (V_{n+j}^h - V_n^h)hq(jh)\Big) \leq 0,$$

so that

$$\left(\frac{1-e^{-\beta\Delta t}}{\Delta t}\right)(V_{n+1}^h - V_n^h) - e^*\Big(\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(V_{n+1+j}^h - V_{n+1}^h) - (V_{n+j}^h - V_n^h)hq(jh)\Big) \leq F\Big((V_{n+1}^h - V_n^h)/h\Big) - F\Big((V_n^h - V_{n-1}^h)/h\Big).$$

Since both terms on the left side of the inequality are nonnegative, we have that

$$F\Big((V_{n+1}^h - V_n^h)/h\Big) - F\Big((V_n^h - V_{n-1}^h)/h\Big) \geq 0,$$

leading to

$$(V_{n+1}^h - V_n^h)/h \leq (V_n^h - V_{n-1}^h)/h$$

as desired. Thus Property 3 is seen to hold.

Property 4. The value functions V^h are bounded by $\frac{\Delta t U(\bar{c})}{1-e^{-\beta \Delta t}}$.

PROOF. This result is an immediate consequence of Property 1 and the proof of Property 2. For sufficiently large i, the optimal exploration rate is 0. Moreover, V^h is increasing, so we have

$$\left(\frac{1-e^{-\beta\Delta t}}{\Delta t}\right)V_n^h = F\Big((V_n^h - V_{n-1}^h)/h\Big) \leq U(\bar{c}).$$

Since V^h is nondecreasing, the result for large i extends to all i.

With these properties in hand, we are now prepared to prove convergence.

4 Convergence Theory for the Discrete Value Function

Our first task in demostrating convergence is to show that, if the sequence converges, then it converges to a viscosity solution of the continuous problem.

Lemma 1. Define the function $v^h(x)$ as the continuous, piecewise linear interpolation of V^h on the interval $[0, \infty)$. Suppose that, for each $B > 0$, $v^h \to v^*$ uniformly on $[0, B]$, as $h \to 0$. Then v^* is a viscosity solution of the Bellman equation (8).

PROOF. Suppose $\psi \in C^2(0, \infty)$, and define

$$A\psi(x) = -\beta\psi(x) + F(\psi'(x)) + \sup_{0\leq e\leq \bar{e}}\Big\{\lambda e\int_0^\infty (\psi(x+y)-\psi(x))q(y)\,dy - He\Big\}.$$

The equation $A\psi = 0$ is a restatement of (8). Likewise, we define

$$\begin{aligned}(A^h\psi)(ih) &= -\left(\frac{1-e^{-\beta\Delta t}}{\Delta t}\right)\psi(ih) + F\Big((\psi(ih)-\psi((i-1)h)/h\Big)\\ &+ \sup_{0\leq e\leq \bar{e}}\Big(e\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(\psi((i+j)h)-\psi(ih))hq(jh) - H\Big)\end{aligned}$$

Note that, due to the C^2 nature of ψ, $A^h\psi$ converges uniformly to $A\psi$ on any compact subset of $(0, \infty)$. To show that v^* is a viscosity subsolution, we need to see that, if x_0 is a strict local maximum of $v^* - \psi$, with $x_0 \in (0, \infty)$, then $A\psi(x_0) \geq 0$. Now, by the uniform convergence of v^h to v^*, there exists a sequence x_h of local maxima of $v^h - \psi$ (necessarily on the grid $\{ih\}$,) converging to x_0. Now, $V^h_{x_h} = T^h(V^h)_{x_h}$, so $\psi(x_h) \le T^h(\psi)_{x_h}$, which leads, by definition of A^h, to $A^h\psi(x_h) \ge 0$. Thus, $A\psi(x_0) \ge 0$, and v^* is a viscosity subsolution. The supersolution argument is identical.

Having shown that any possible limit must be the "right" limit, we now turn to the problem of showing that a limit must exist. Toward that end, we require the following assumption.

(†) The function U is nondecreasing, concave, and continuously differentiable on $[0, \bar{c}]$, with $U(0) = 0$ and $U'(0) = p > 0$.

Lemma 2. Define the function $v^h(x)$ as the continuous, piecewise linear interpolation of V^h on the interval $[0, \infty)$. Suppose that U satisfies (†), and that $\beta > \lambda\bar{e}\mu_q$. Then for B, v^h is equicontinuous on $[0, B]$.

PROOF. First,we set $\varepsilon = \beta - \lambda\bar{e}\mu_q > 0$. Next, we choose h_0 so that if $0 < h < h_0$, then

$$\begin{aligned} |\sum_{j=1}^{\infty}(jh)q(jh)h - \mu_q| &< \frac{\varepsilon}{4\lambda\bar{e}}, \\ e^{-\beta\Delta t}\Big(1 + \Delta t\lambda\bar{e}\mu_q) \le 1 - \varepsilon/2, \end{aligned}$$

Next, for each $h \in (0, h_0)$, choose N_h so that

$$|\sum_{j=1}^{N_h}(jh)q(jh)h - \sum_{j=1}^{\infty}(jh)q(jh)h| < \frac{\varepsilon}{4\lambda\bar{e}}.$$

Now, we define the functions w^h by interpolating linearly on the points between the grid ih, with $w^h(ih) = 2pih$, when $i \le N_h$, and $w^h(ih) = w^h(N_h h)$ when $i > N_h$. This function is in $\mathcal{V}$. Moreover, because $U'(c) \le p$,, we must have, for $i \le N_h$, that

$$\begin{aligned} T^h(w^h)(ih) = &= e^{-\beta\Delta t}w^h(ih) + \Delta t \sup_{0\le e\le\bar{e}}\Big\{e\Big(\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(V^h_{i+j} - V^h_i)hq(jh) - H\Big)\Big\} \\ &\le e^{-\beta\Delta t}\Big(2pih + 2p\lambda\bar{e}(\mu + \frac{\varepsilon}{2\lambda\bar{e}}\Big) \end{aligned}$$

For $i = 1$, we have that $T^h(w^h)(h) \le w^h(h)$, so by the contraction principle, we must have $w^h(h) \ge V^h_1$. Since $V^h_0 = 0$, and since V^h is nondecreasing and concave, we have a bound on the derivative of v^h for sufficiently small h: $v^h(x) \leq 2p$, for all x.

Appealing to the Arzela-Ascoli theorem, we have the following convergence result:

Theorem 1. Under the conditions of Lemma 2, the sequence v^h converges to v^*, the unique viscosity solution of (8). The convergence is uniform each compact interval $[0, B]$.

An argument identical to that of Lemma 5 of [7] allows us to claim that $v^{h\prime}(x)$ converges to $v^{*\prime}(x)$, uniformly on compact sets $[0, B]$. That argument also allows us to claim that the discrete feedback controls converge to the feedback controls of the continuous problem, as well. In particular, the feedback function c^h converges uniformly to the optimal feedback consumption rate c on each compact set $[0, B]$. The discontinuity of the linear optimization of the e term does not allow a uniform convergence result, but we can obtain the result that $e^h(x) \to e(x)$ for each x at which the feedback exploration e is continuous. The distinction between the approach here and the general approach of [13] is that the particular form of the problem here allows us to derive these tighter results concerning derivative and control convergence.

Having developed a convergence theory for the approximations, we now consider some numerical examples.

5 Computations and Some Robust Alternatives

In order to illustrate our results, we present results from numerical simulation based on the scheme described above. The basic idea here is to examine the behavior of the optimal controls in the presence of various perturbations to the system.

Recall that our basic dynamic model is given by

$$dX(t) = dI_t - c_t\, dt,$$

in which I is defined as the discovery jump process, whose transitions are given by

$$Pr[I(t+h) = y | I(t) = y, e] \quad = \quad 1 - \lambda e h + o(h)$$

$$Pr[I(t+h) \in (y+a, y+b] | I(t) = y, e] \;=\; \lambda e h \int_a^b q(z)\, dz + o(h).$$

The variables c and e are the control variables: $c(t) \in [0, \bar{c}], e(t) \in [0, \bar{e}]$ are to be chosen in an optimal manner over time.

The dynamics, as discussed in 2, are discretized into a Markow chain structure. The continous resource level is discretized onto a grid ih, for $i = 0, 1, 2, \cdots$, and the ability to approximate closely is governed by the step h. Time is discretized in a related manner: $\Delta t = h^2/Q$, where $Q = \bar{c}h + \lambda \bar{e} h^3 \sum_{j=1}^{\infty} q(jh)$. The transition probabilities of this chain are given by

$$\begin{aligned}
P^{c,e}_{i,i-1} &= \frac{c\Delta t}{h}, \\
P^{c,e}_{i,i+j} &= \lambda e \Delta t h q(jh), \quad j \geq 1, \\
P^{c,e}_{i,i} &= 1 - P^{c,e}_{i,i-1} - \sum_{j=1}^{\infty} P^{c,e}_{i,i+j},
\end{aligned}$$

in which $\Delta t = h^2/Q$, with the denominator Q chosen to ensure all quantities are between 0 and 1. In particular, we choose Q as

$$Q = \bar{c}h + \lambda \bar{e} h^3 \sum_{j=1}^{\infty} q(jh).$$

The scheme we implement involves several differences from the scheme analyzed above. In particular, we must truncate the resource level at a maximum value, in order to be able to compute the value function V^h. Toward that end we specific N to be the maximal discrete state, and we set, for each i

$$P^{c,e}_{i,N} \;=\; \lambda e \Delta t \sum_{j=N-i}^{\infty} h q(jh)$$

to provide an upper limit. The model has the effect of limiting our ability to “store” the resource: for any discovery that brings our resource level to a value greater than Nh, we “lose” the overflow.

With the transition probabilities defined, we construct the Bellman equation

$$V^h_i = T^h(V)_i = \sup_{c,e} \Big\{ e^{-\beta \Delta t} \sum_{j=0}^{N} P^{c,e}_{i,j} V^h_j + \Delta t G(c, e, ih) \Big\},$$

in which the cost functional to be maximized is

$$J(ih,c,e) = E\Big[\sum_{n=1}^{\infty} e^{-\beta n\Delta t} G(c_{n\Delta t}, e_{n\Delta t}, X_{n\Delta t})\Big].$$

The reason for the general form G over the form analyzed above is to examine several possible perturbations away from our assumptions.

For the objective functional, our baseline choice is a utility function defined by a Hyperbolic Absolute Risk Aversion (HARA) form:

$$G(c,e,x) = U(c) - He, \qquad U(c) = Ac^{\gamma},$$

in which $0 < \gamma < 1$. These functions (as discussed in [1, 4]) are chosen so that increasing rates of consumption produce decreasing marginal utility. In the case of the HARA utility ($\gamma < 1$), the marginal utility at 0 consumption rate is infinite, meaning intuitively that there is a huge gain in utility for small increments in consumption rate near zero. As consumption rates increase, the additional utility of a small increment in consumption rate is less. To contrast, a linear utility function is considered risk neutral, while a convex utility function ($\gamma > 1$) models a risk-seeking individual.

Also, we manipulate the cost with some alternatives meant to make the controls behave in a risk averse manner. What we mean by this statement is that we seek cost functionals that will produce controls less likely to drive the system to 0 resource stock.

The first adjustment to the cost function we consider is a function of the state of HARA form:

$$G(c,e,x) = U(c(t)) - He(t) + k(X(t))$$

in which $k(x)$ adds value, especially at low levels of resource stock. We use $k(x) = b\sqrt{x} - dx$ which includes a holding cost for storing the resource and a utility for having a stock. At small x, the $\sqrt{x}$ term dominates, while for large x the holding cost is more important. The analysis of [16] includes the holding cost, but not the HARA-like $\sqrt{x}$ term.

Another possible adjustment of the cost is

$$G(c,e,x) = U(c,x) - He = Ac - b\frac{c^2}{x} - He,$$

as suggested by Pindyck in [15]. The state dependent utility tends to reduce consumption when the resource is low.

Note that each of these choices decouples the exploration variable from the consumption variable. In each of these cases, we can write $G(c, e, x) = U(c, x) - He$. This structure simplifies the Bellman equation significantly. Indeed, the operator T^h is easily seen to have the form

$$\begin{aligned} T^h(V^h)_i &= e^{-\beta\Delta t}V_i^h + \Delta t \sup_{0\le c\le \bar{c}} \Big\{U(c, ih) - e^{-\beta\Delta t} c \frac{V_i^h - V_{i-1}^h}{h}\Big\} \\ &+ \Delta t \sup_{0\le e\le \bar{e}} \Big\{ e\Big(\lambda e^{-\beta\Delta t} \sum_{j=1}^{\infty} (V_{i+j}^h - V_i^h) hq(jh) - H\Big)\Big\} \end{aligned}$$

Given a candidate value function V^h, one can determine analytically the choice of c and of e for each i. The choice of c will depend on the form and structure of U, but the simple HARA or polynomial forms proposed herein admit straightforward computation. The choice of e is even simpler: either $e = 0$ or $e = \bar{e}$, depending on the sign of the quantity

$$\lambda e^{-\beta\Delta t} \sum_{j=1}^{\infty} (V_{i+j}^h - V_i^h) hq(jh) - H.$$

Now, in order to solve the dynamic programming equation, there are two common approaches (as discussed in detail in Volume 2 of [2]). One, value iteration, makes direct use of the contraction principle. Starting with an initial guess for the value function, W^0, we construct a sequence $W^{n+1} = T^h(W^n)$, which is guaranteed to converge to the unique fixed point of T^h. The convergence rate, however, is governed by $e^{-\beta\Delta t}$, and as Δt becomes small, the convergence rate is too slow for practical use. The second approach, policy iteration, begins with an initial guess of the policies, c^0 and e^0. Given policies c^n and e^n, we perform two steps. First, we solve the system of linear equations

$$\begin{aligned} W_i^n &= e^{-\beta\Delta t}W_i^n + \Delta t \Big\{U(c, ih) - e^{-\beta\Delta t} c \frac{W_i^n - W_{i-1}^n}{h}\Big\} \\ &+ \Delta t \Big\{ e\Big(\lambda e^{-\beta\Delta t} \sum_{j=1}^{\infty} (W_{i+j}^n - W6n_i) hq(jh) - H\Big)\Big\} \end{aligned}$$

for W^n. Having computed W^n, we then determine the next policies in the iteration, c^{n+1} and e^{n+1} by maximizing

$$e\left(\lambda e^{-\beta\Delta t}\sum_{j=1}^{\infty}(W^n_{i+j}-W^n_i)hq(jh)-H\right).$$

and

$$\left(U(c,ih)-e^{-\beta\Delta t}c\frac{W^n_i-W^n_{i-1}}{h}\right).$$

Policy iteration, a reformulation of Newton's method for solving the Bellman equation, tends to converge much more quickly that value iteration, at least in problems for which the linear system of equations can be solved quickly.

In the examples below, we implemented the Markow chain in Matlab, using the policy iteration scheme to compute the value and policies. The following parameter values were used to illustrate the approach.

Parameter	Value
β	1
H	0.1
λ	1
q	$2e^{-2x}$
$\bar{c}$	10.0
$\bar{e}$	10.0
h	0.01
N	500

The "regular" utility of consumption we use is $U(c)=\sqrt{c}$. The robust adjustment we add is $k(x)=\sqrt{x}-x/2$. The Pindyck utility we use is $U(c,x)=c-c^2/(10x)$. In Figures 1 and 2, we see the consumption rates and the exploration rates for these three cost functionals.

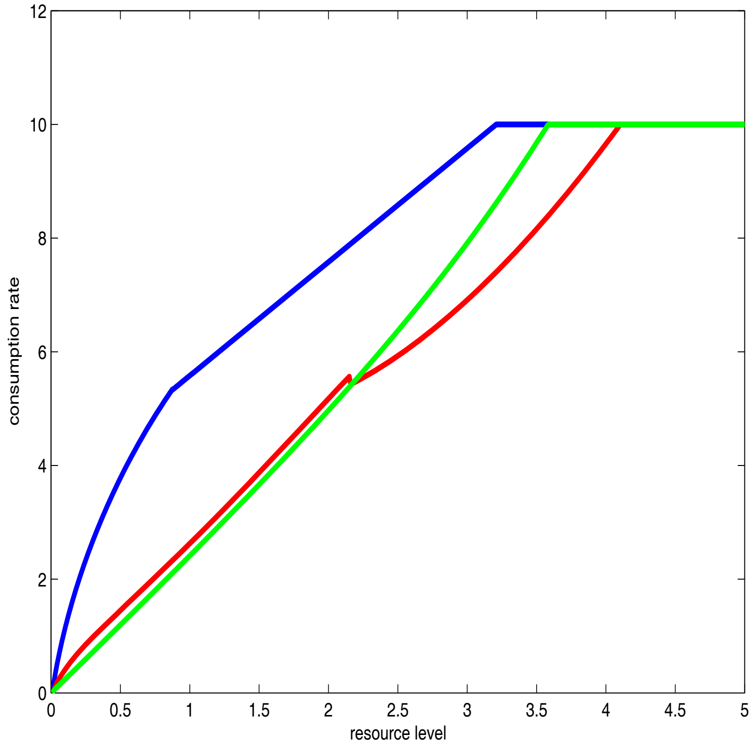

Figure 1: Optimal consumption as function of resource

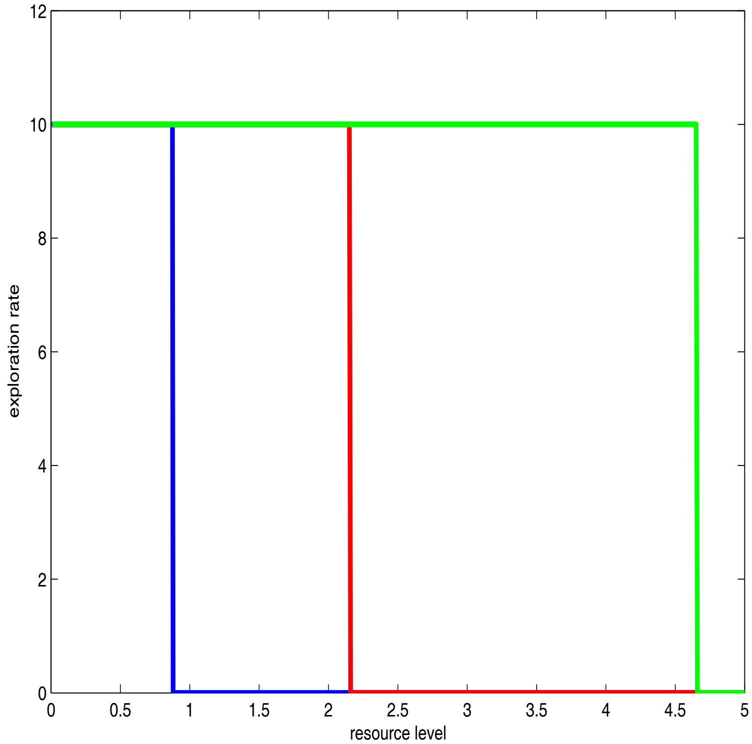

Figure 2: Optimal exploration as function of resource

In these two figures, one can see that the consumption rates for the standard HARA, the risk-averse stocking utility, and the Pindyck risk-averse model are all rather different from each other. Moreover, the Pindyck utility recommends exploration for a larger range of stock levels. In order to see if these different objectives result in different resource levels, we developed a Monte Carlo simulation. Using the same realization on each model (but using the transition probabilities for each control strategy), we simulated 20,000 time steps of the chain. In Figure 3, we see the three trajectories simulated. We can see the effects of reduced consumption as well as the effects of additional exploration. The realizations of the risk averse control schemes keep the stock away from zero.

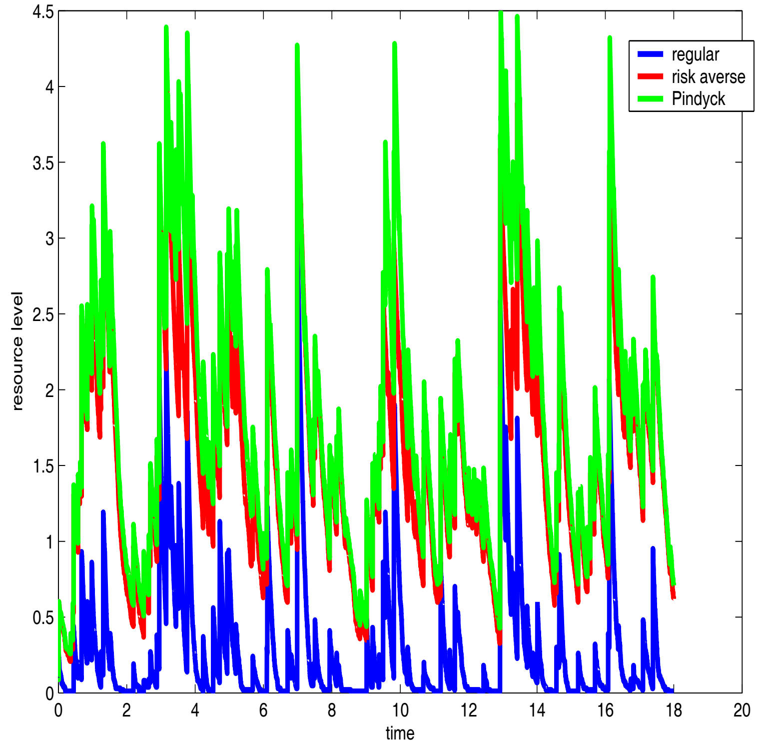

Figure 3: Optimal exploration as function of resource

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