Weak Lie 2-algebras

In HDA6, we introduced the notion of a *semistrict Lie 2-algebra*. As suggested by its name, a semistrict Lie 2-algebra is not the completely weakened version of Lie algebra. Here, however, we further weaken the skew-symmetry of the bracket to obtain a *weak Lie 2-algebra*. We first remind the reader of the notions of n-linear functors and natural transformations:

Definition 1. *Let V and W be 2-vector spaces. A functor $F\colon V^n \to W$ is* ***n*-linear** *if $F(x_1, \dots, x_n)$ is linear in each argument, where $(x_1, \dots, x_n)$ is an object or morphism of V^n. Given n-linear functors $F, G\colon V^n \to W$, a natural transformation $\theta\colon F \Rightarrow G$ is* ***n*-linear** *if $\theta_{x_1,\dots,x_n}$ depends linearly on each object x_i.*

Definition 2. *A* **weak Lie 2-algebra** *consists of:*

- *a 2-vector space L*

equipped with

- *a bilinear functor, the* **bracket**, $[\cdot,\cdot]\colon L \times L \to L$
- *a trilinear natural isomorphism, the* **Jacobiator**,

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y]$$

- *a bilinear natural isomorphism, the* **Antisymmetrizer**,

$$A_{x,y}\colon [x,y] \to -[y,z],$$

which are required to satisfy

- *the* **Jacobiator identity**:

$$J_{[w,x],y,z} \circ [J_{w,x,z}, y] \circ (J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) =$$
$$[J_{w,x,y}, z] \circ (J_{[w,y],x,z} + J_{w,[x,y],z}) \circ [J_{w,y,z}, x] \circ [w, J_{x,y,z}]$$

- $$A_{[x,y],z} \circ [z, -A_{x,y}] = [A_{x,y}, z] \circ -A_{[y,z],z}$$

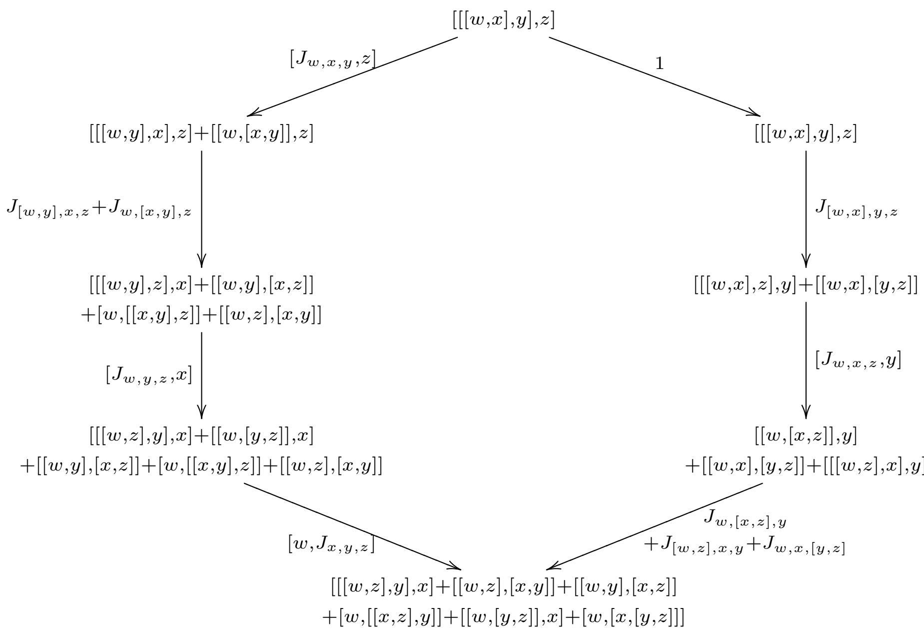
- $$J_{x,y,z} \circ J_{x,z,y} \circ [x, A_{y,z}] = 1_{[[x,y],z]}$$

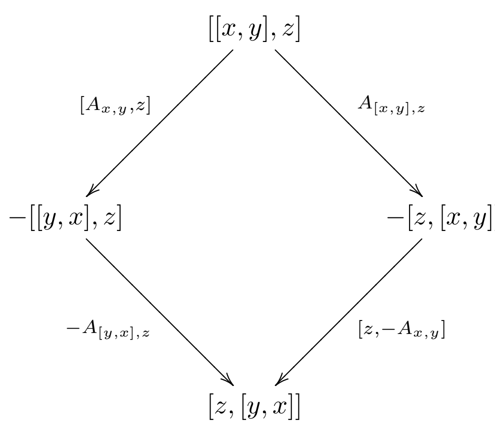
- $$[A_{x,y}, z] \circ -J_{y,x,z} = J_{x,y,z} \circ A_{[x,z],y} \circ A_{x,[y,z]}$$

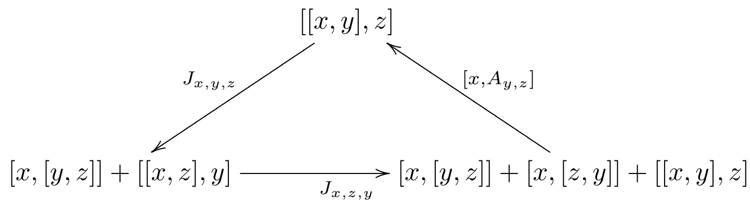
- $$J_{x,y,z} \circ A_{x,[y,z]} \circ J_{x,z,y} \circ [x, A_{z,y}] \circ -A_{x,[y,z]} = 1_{[[x,y],z]}$$

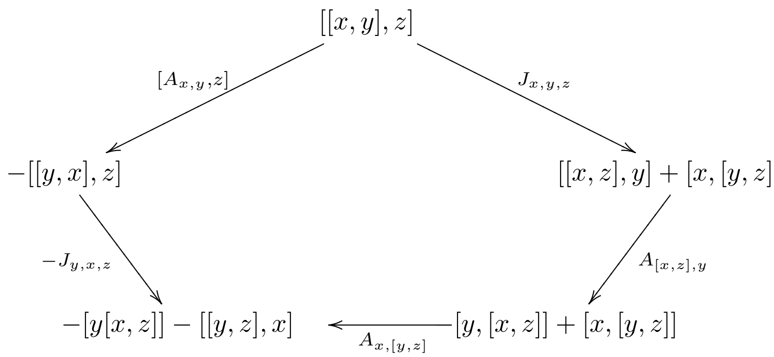
for all $w, x, y, z \in L_0$.

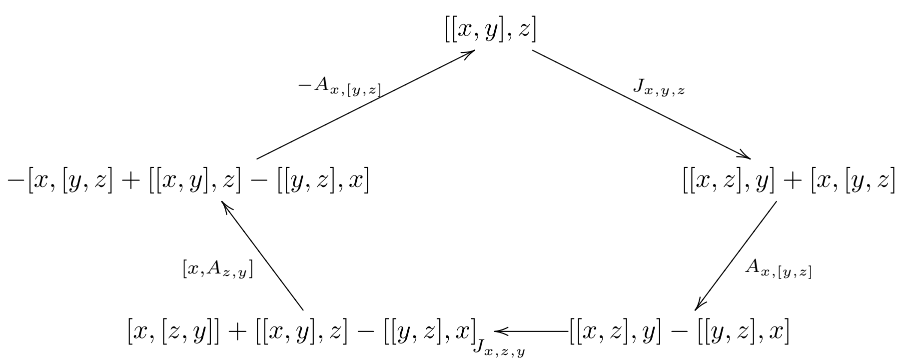
In their algebraic formulations, these coherence laws are not very illuminating. Drawing them as commutative diagrams, we see that they






$[[x,y],z]$
$J_{x,y,z}$
$[x,A_{y,z}]$
$[x,[y,z]]+[[x,z],y]$
$J_{x,z,y}$
$[x,[y,z]]+[x,[z,y]]+[[x,y],z]$


$[[x,y],z]$
$[A_{x,y},z]$
$J_{x,y,z}$
$-[[y,x],z]$
$[[x,z],y]+[x,[y,z]]$
$-J_{y,x,z}$
$A_{[x,z],y}$
$-[y[x,z]]-[[y,z],x]$
$A_{x,[y,z]}$
$[y,[x,z]]+[x,[y,z]]$


$[[x,y],z]$
$-A_{x,[y,z]}$
$J_{x,y,z}$
$-[x,[y,z]+[[x,y],z]-[[y,z],x]$
$[[x,z],y]+[x,[y,z]]$
$[x,A_{z,y}]$
$A_{x,[y,z]}$
$[x,[z,y]]+[[x,y],z]-[[y,z],x]$
$J_{x,z,y}$
$[[x,z],y]-[[y,z],x]$