

Lie 2-algebras

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Abstract

We categorify the theory of Lie algebras beginning with a new notion of categorified vector space, or ‘2-vector space’, which we define as an internal category in \mathbf{Vect} , the category of vector spaces. We then define a ‘semistrict Lie 2-algebra’ to be a 2-vector space L equipped with a skew-symmetric bilinear functor $[\cdot, \cdot]: L \times L \rightarrow L$ satisfying the Jacobi identity up to a completely antisymmetric trilinear natural transformation called the ‘Jacobiator’, which in turn must satisfy a certain law of its own. We construct a 2-category of semistrict Lie 2-algebras and show that it is 2-equivalent to the 2-category of ‘2-term L_∞ -algebras’ in the sense of Stasheff. We also investigate strict and skeletal Lie 2-algebras. We show how to obtain the strict ones from Lie 2-groups and we use the skeletal ones to classify Lie 2-algebras in terms of 3rd cohomology classes in Lie algebra cohomology. This classification allows us to construct for any finite-dimensional Lie algebra \mathfrak{g} a canonical 1-parameter family of Lie 2-algebras \mathfrak{g}_\hbar which reduces to \mathfrak{g} at $\hbar = 0$.

We then explore the relationship between Lie algebras and algebraic structures called ‘quandles’. A quandle is a set Q equipped with two binary operations $\triangleright: Q \times Q \rightarrow Q$ and $\triangleleft: Q \times Q \rightarrow Q$ satisfying axioms that capture the essential properties of the operations of conjugation in a group and algebraically encode the three Reidemeister moves. Indeed, we describe the relation to groups and show that quandles give invariants of braids. We further show that both Lie algebras and quandles give solutions of the Yang–Baxter equation, and explain how conjugation plays a prominent role in the both the theories of Lie algebras and quandles. Inspired by these commonalities, we provide a novel, conceptual passage from Lie groups to Lie algebras using the language of quandles. Moreover, we propose relationships between higher Lie theory and higher-dimensional braid theory. We conclude with evidence of this connection by proving that any semistrict Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation, which is the higher-dimensional analog of the Yang–Baxter equation.