# RESEARCH SUMMARY

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## 1. Introduction

Higher-dimensional algebra is the study of generalizations of algebraic concepts obtained through a process called 'categorification'. My research in higher-dimensional algebra develops and explores categorified Lie algebras, also called Lie 2-algebras.

In the mid-1990's, Crane [C, CF] coined the term categorification to refer to the process of developing category-theoretic analogs of set-theoretic concepts. In this process we replace elements with objects, sets with categories, and functions with functors. We replace equations between elements by isomorphisms between objects, and replace equations between functions by natural isomorphisms between functors. Finally, we require that these isomorphisms satisfy equations of their own, called coherence laws. Finding the correct coherence laws is often the most difficult aspect of this generalization process. Ultimately, by iterating this process, mathematicians wish to obtain and apply the $n$-categorical generalizations of as many mathematical concepts as possible to strengthen and simplify the connections between different subfields of mathematics.

Perhaps the greatest strength of categorification is that it allows us to refine our concept of 'sameness' by enabling us to distinguish between equality and isomorphism. In a set, two elements are either the same or different, while in a category, two objects can be isomorphic, but not equal. This more careful consideration of the notion of sameness is the reason that categorification plays an increasingly important role not only in mathematics, but also in physics and computer science, where a precise treatment of the notion of sameness is crucial.

## 2. Lie 2-algebras

A Lie 2-algebra blends the notion of a Lie algebra with that of a category. Just as a Lie algebra has an underlying vector space, a Lie 2-algebra has an underlying 2-vector space. A 2-vector space is a hybrid of the notions of vector space and category. That is, it is a category where everything is *linear*. More precisely, a **2-vector space** $V$ is a category consisting of vector spaces $V_0$ and $V_1$ of objects and morphisms, respectively, together with linear source and target maps $s, t\colon V_1 \to V_0$, a linear identity-assigning map $i\colon V_0 \to V_1$, and a linear composition map $\circ\colon V_1 \times_{V_0} V_1 \to V_1$. My recent paper with John Baez [BC] contains a development of the theory of 2-vector spaces. This new theory of 2-vector spaces has already begun to play a role in the representation theory of categorified groups, or 2-groups, as well as in topological quantum field theory [E,G,P].

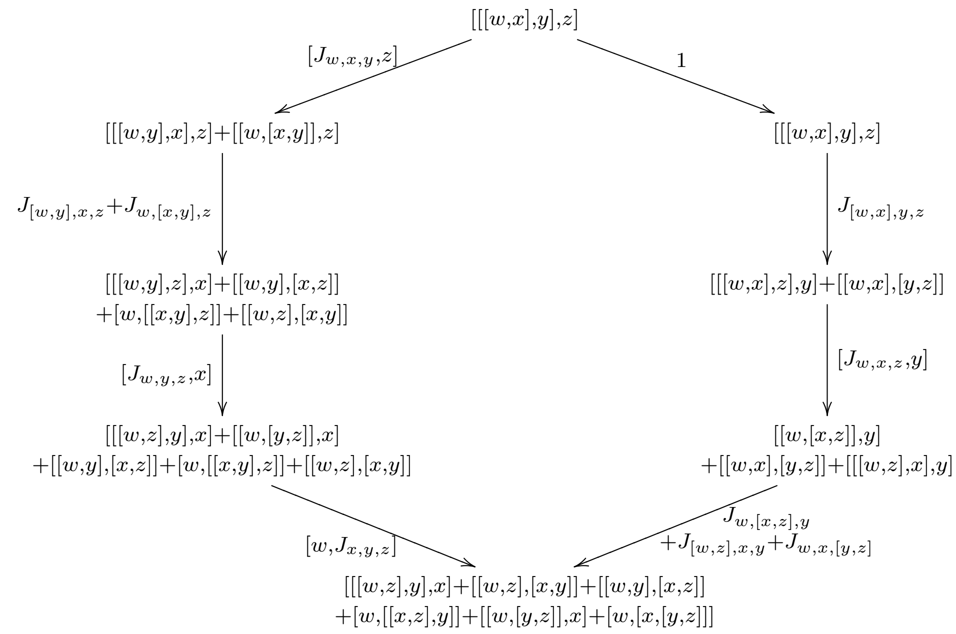
To obtain a **Lie 2-algebra**, we begin with a 2-vector space $L$ and equip it with a bilinear, skew-symmetric bracket *functor*

$$[\cdot,\cdot]\colon L \times L \to L,$$

which satisfies the Jacobi identity *up to a natural isomorphism* called the 'Jacobiator',

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

The Jacobiator is completely antisymmetric and trilinear and satisfies a law called the 'Jacobiator identity', which says this octagon commutes:

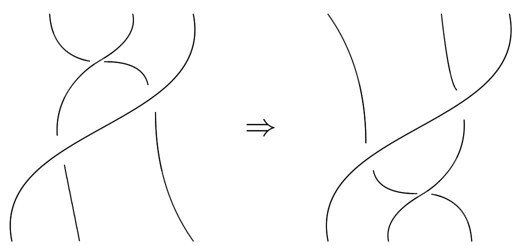



for all $w, x, y, z \in L_0$. This diagram expresses that the two ways of using the Jacobiator to rebracket the expression $[[[w, x], y], z]$ are the same.

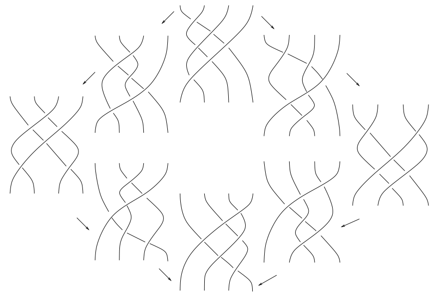
## 3. Relation to Topology

The coherence law for the Jacobiator is intimately related to the Zamolodchikov tetrahedron equation. This equation plays a role in the theory of knotted surfaces in 4-space analogous to that played by the Yang–Baxter equation, or third Reidemeister move, in the theory of knots in 3-space. Since the algebraic version of this equation is not particularly illuminating, we offer a geometric description instead.

Consider the surface in 4-space traced out by the *process of performing* the third Reidemeister move:



The Zamolodchikov tetrahedron equation says the surface traced out by first performing the third Reidemeister move on a threefold crossing and then sliding the result under a fourth strand is isotopic to that traced out by first sliding the threefold crossing under the fourth strand and then performing the third Reidemeister move. The following commutative octagon formalizes this process:



Any Lie algebra gives a solution of the Yang–Baxter equation. In fact, under suitable conditions, the Yang–Baxter equation and Jacobi identity are actually equivalent. My dissertation contains a higher-dimensional analog: Any Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation. That is, under suitable conditions, the Zamolodchikov tetrahedron equation is equivalent to the Jacobiator identity.

## 4. Relation to Lie Algebra Cohomology

Since the correct notion of sameness for categories is equivalence rather than isomorphism, the same is true for Lie 2-algebras. In my dissertation, I classified Lie 2-algebras up to equivalence in terms of third cohomology classes in Lie algebra cohomology. I have shown that there is a one-to-one correspondence between equivalence classes of Lie 2-algebras and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space $V$, a representation $\rho$ of $\mathfrak{g}$ on $V$, and an element of $H^3(\mathfrak{g}, V)$. This result provides a new interpretation of $H^3(\mathfrak{g}, V)$ in terms of Lie 2-algebras.

One of the more interesting examples of a finite-dimensional Lie 2-algebra, characterized in terms of the quadruple described above, consists of: a finite-dimensional Lie algebra $\mathfrak{g}$ over the field $k$, a vector space $k$, the trivial representation $\rho$ of $\mathfrak{g}$ on $k$, and the 3-cocycle $\langle x, [y, z]\rangle$ where $\langle \cdot, \cdot \rangle$ is the Killing form.

In fact, every finite-dimensional Lie algebra $\mathfrak{g}$ admits a one-parameter deformation $\mathfrak{g}_\hbar$ in the category of Lie 2-algebras by choosing the 3-cocycle $\hbar\langle x, [y, z]\rangle$ where $\hbar$ is any element of $k$. I suspect that such Lie 2-algebras have connections to quantum groups and affine Lie algebras and intend to pursue this relationship further.

## 5. Current and Future Work

Since theories of 2-groups and Lie 2-groups already exist [BLau], a natural question is whether Lie 2-algebras arise from Lie 2-groups. In this direction, I am developing categorified quandles, called 2-quandles, so that I may categorify the passage from Lie groups to Lie algebras relying on the language of quandles, which I introduced in my dissertation. In the process of obtaining Lie 2-algebras from Lie 2-groups, I expect to demonstrate that 2-quandles provide invariants of the 2-braids introduced by Carter and Saito [CS]. Furthermore, I am also investigating how to replace the equation $[x, y] = -[y, x]$ by an isomorphism in order to formulate a definition of a weak Lie 2-algebra. Thus far, it appears such Lie 2-algebras possess five new coherence laws in addition to the Jacobiator identity.

Moreover, I am exploring the representation theory of the Lie 2-algebras $\mathfrak{g}_\hbar$ together with John Baez and Danny Stevenson. Preliminary results indicate another relationship to Lie algebra cohomology. After completing these projects, I hope to categorify other ideas in differential geometry, such as the exponential map, and to determine whether or not a homomorphism between Lie 2-groups induces a homomorphism between the corresponding weak Lie 2-algebras.

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