Quandle Queries

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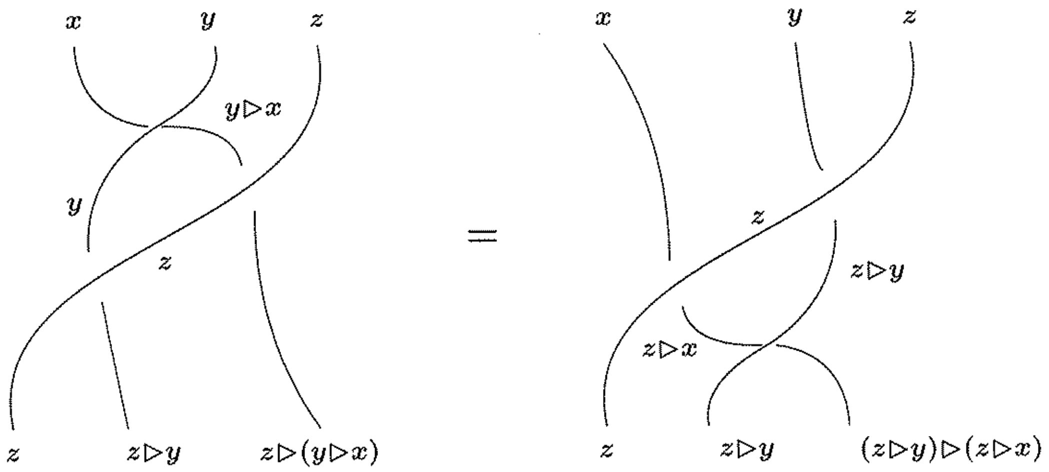


Knots in Washington XXIII

November 17, 2006

Definitions

A **quandle** $(Q, \triangleright, \triangleleft)$ is a set Q equipped with two binary operations

$$\triangleright : Q \times Q \to Q \quad \text{and} \quad \triangleleft : Q \times Q \to Q$$

that satisfy for all $x, y, z \in Q$:

(i) $x \triangleright x = x$

(ii) $x \triangleright (y \triangleleft x) = y = (x \triangleright y) \triangleleft x$

(iii) $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$

This is equivalent to the more familiar definition:

A **quandle** $(Q, \triangleleft)$ is a set Q equipped with a binary operation

$$\triangleleft : Q \times Q \to Q$$

that satisfies:

(a) $x \triangleleft x = x$ for all $x \in Q$

(b) For all $x, y \in Q$, there exists a unique $z \in Q$ such that $z \triangleleft x = y$

(c) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ for all $x, y, z \in Q$

A **rack** $(Q, \lhd)$ is a set Q equipped with a binary operation

$$\lhd \colon Q \times Q \to Q$$

that satisfies:

(b) For all $x, y \in Q$, there exists a unique $z \in Q$ such that $z \lhd x = y$

(c) $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$ for all $x, y, z \in Q$

Examples

- **Groups** Let G be a group. G is a quandle with

  $$a \rhd b = aba^{-1} \text{ and } a \lhd b = b^{-1}ab$$

  for $a, b \in G$

- **Subgroups** Let G be a group and H a subgroup. Let s be an automorphism of G that fixes the elements of H. G/H is a quandle with

  $$Ha \lhd Hb = Hs(ab^{-1})b$$

  for $a, b \in G$

- **Linear Algebra** Let T be an invertible linear transformation on a vector space V. V is a quandle with

  $$x \lhd y = T^{-1}(y - x) + y$$

  for $x, y \in V$

- **Alexander Quandle** A $\mathbb{Z}([t, t^{-1}])$-module M is a quandle with

$$a \lhd b = ta + (1-t)b$$

for $a, b \in M$

- **Reflection Quandle** $\mathbb{R}^2$ is a quandle with

$$a \rhd b = b \lhd a = \text{“reflect b about a”} = 2a - b$$

for $a, b \in \mathbb{R}^2$

- **Dihedral Quandle** $\mathbb{Z}_n$ is a quandle with

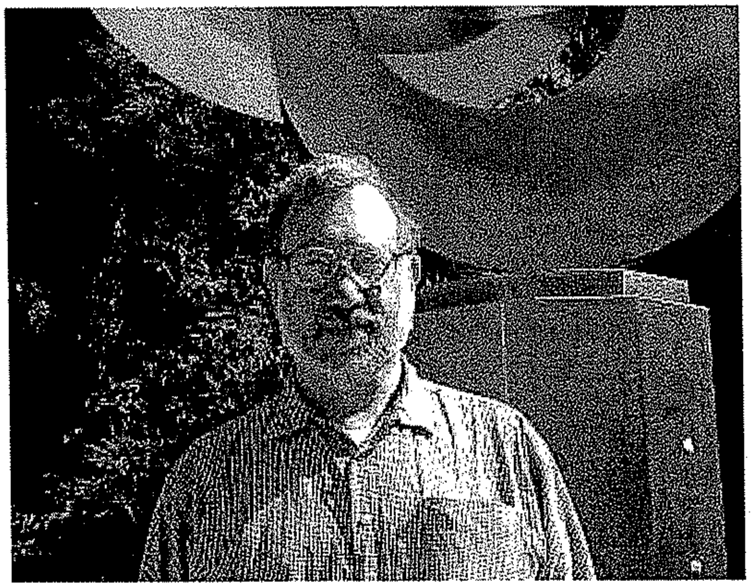
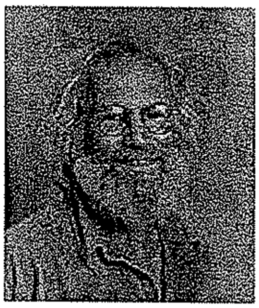
$$i \lhd j \equiv 2j - i \pmod{n}$$

for $i, j \in \mathbb{Z}_n$

History

- Unpublished correspondence between Conway and Wraith in 1960s
- Brieskorn, Dehornoy, Fenn, Ghoneim, Joyce, Kauffman, Kepka, Matveev, Rourke, Takasaki



History

- Automorphic sets, keis, distributive groupoids, crystals

- Word *quandle* due to Joyce

- Classification work done by Graña and Nelson

Coloring of Knots

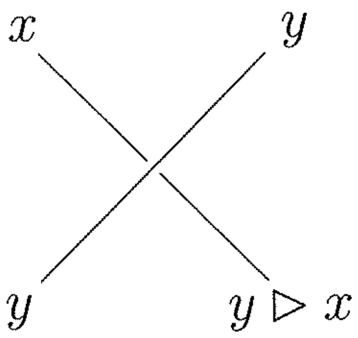
Given a quandle Q define a map

$$B : Q \times Q \to Q \times Q \text{ as}$$

$$B(x, y) = (y, y \triangleright x)$$

Pictorially, B takes the form:

$B :$



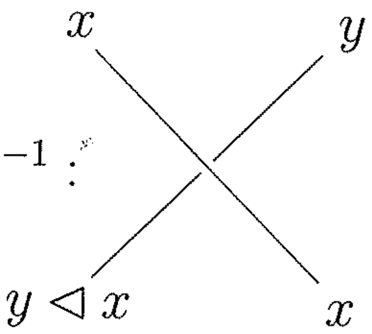
Additionally define

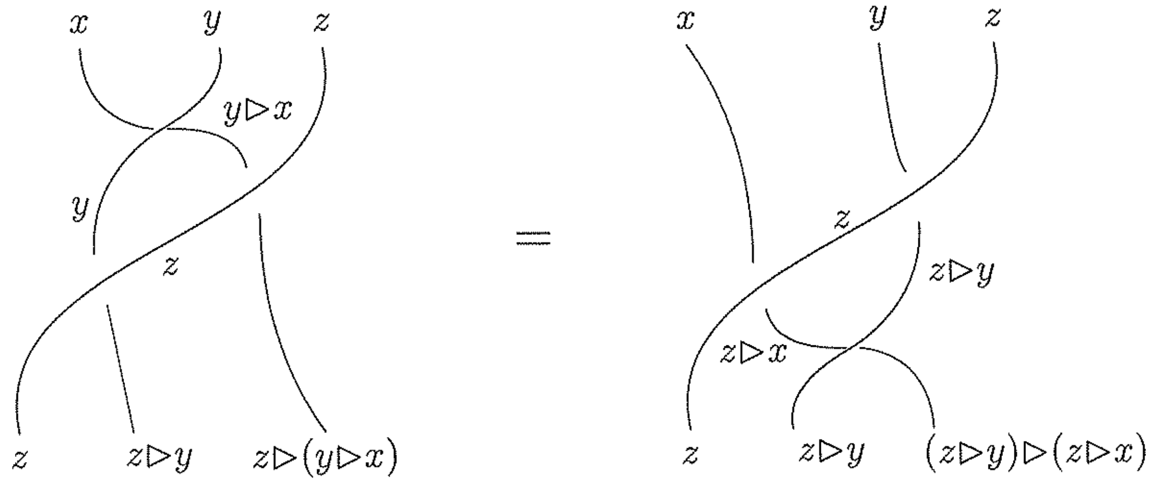
$$B^{-1} : Q \times Q \to Q \times Q \text{ as}$$

$$B^{-1}(x, y) = (y \triangleleft x, x)$$

Pictorially, B^{-1} takes the form:

$B^{-1} :$

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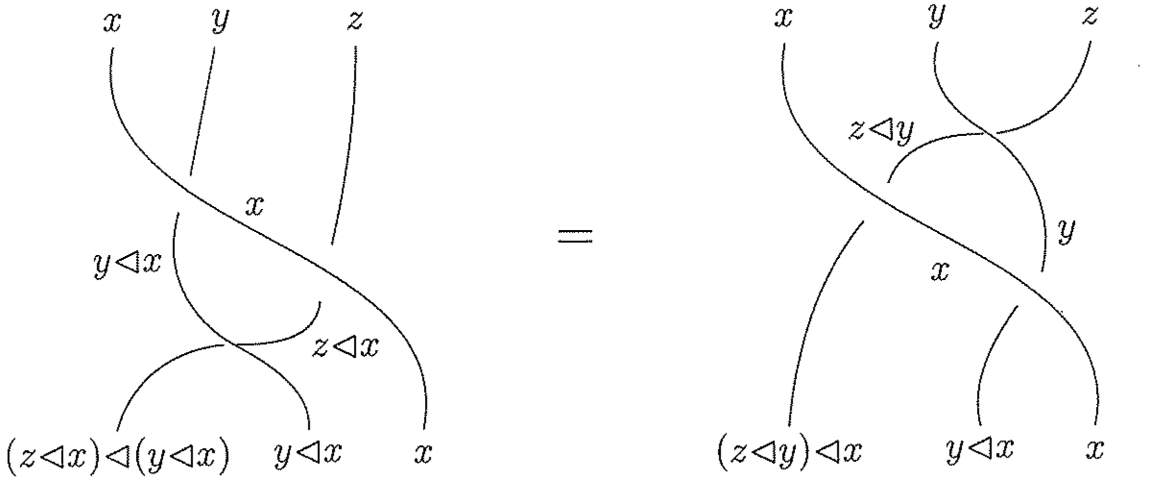
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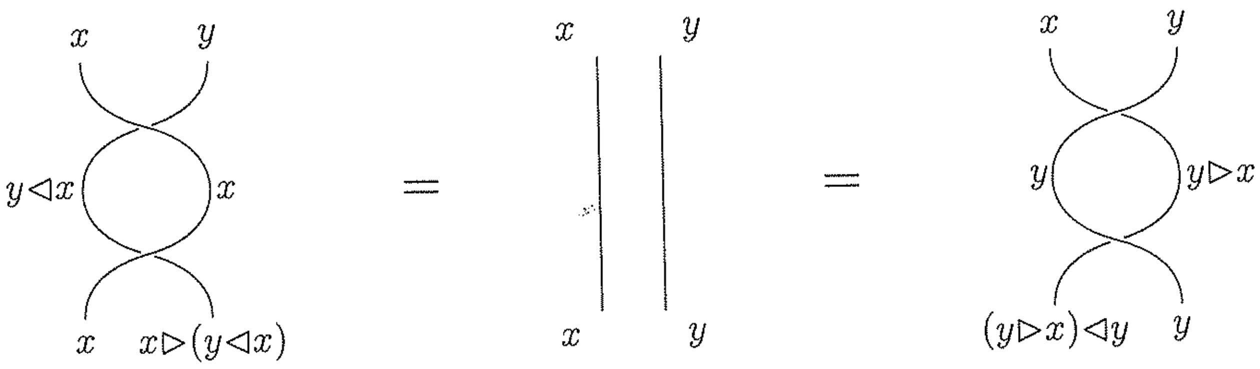
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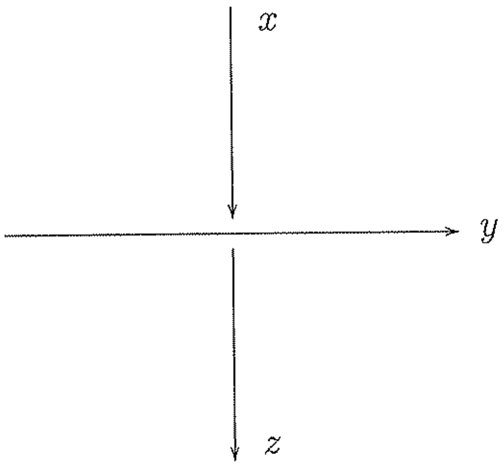
Knot Quandles

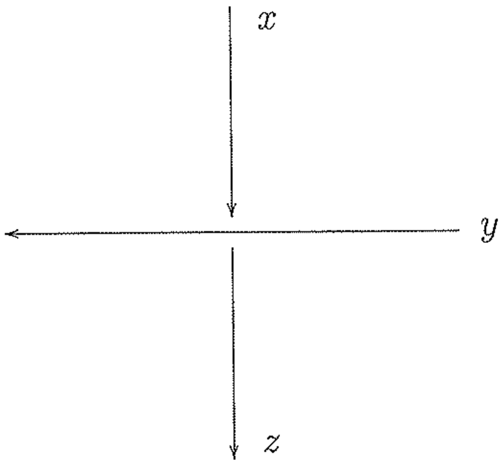
Consider an oriented projection of a knot K.

Step 1: Label the arcs of K by $a, b, c, \dots$

Step 2: At each underpass, read off a relation on the arcs according to the following conventions:



$x \lhd y = z$



$y \rhd x = z$

Let $Q(K)$ be the quandle generated by the arcs with relations given by the crossings.

Cohomology

- For racks defined by Etingof and Graña
- For quandles defined by Carter, Jelsovsky, Kamada, Langford, and Saito
- Later explored by Ameur, Carter, Elhamdadi, Graña, Harris, Mochizuki, Nikiforou, Preygel, Rose, Saito, and Smuddle

A quandle 2-**cocycle** is a linear function ϕ satisfying

$$\phi(x,y) - \phi(x,z) + \phi(x \lhd y, z) - \phi(x \lhd z, y \lhd z) = 0$$

$\forall x, y, z \in X$ and $\phi(x,x) = 0$ for all $x \in X$.

A quandle 3-**cocycle** is a function θ satisfying

$$\theta(x,y,z) + \theta(x \lhd z, y \lhd z, w) + \theta(x,z,w) =$$

$$\theta(x \lhd y, z, w) + \theta(x,y,w) + \theta(x \lhd w, y \lhd w, z \lhd w)$$

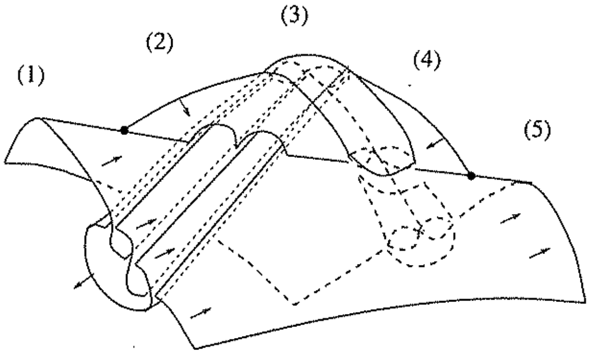
and

$$\theta(x,x,y) = \theta(x,y,y) = 0$$

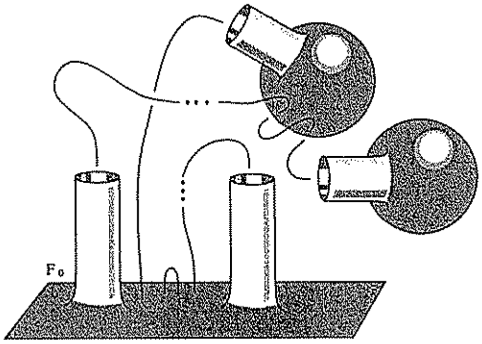
for all $x, y, z, w \in X$.

Applications

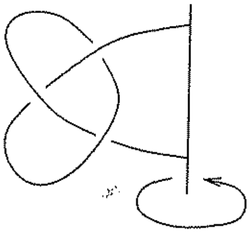
- Non-invertibility of knotted surfaces
- Minimal number of triple points on knotted surface projections (Satoh, Shima, Hatakenaka)



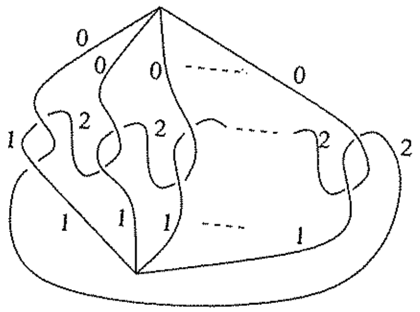
- Ribbon concordance



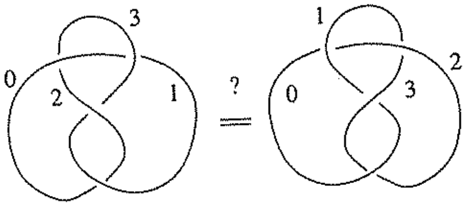
- Minimal number of broken sheets in knotted surface diagram



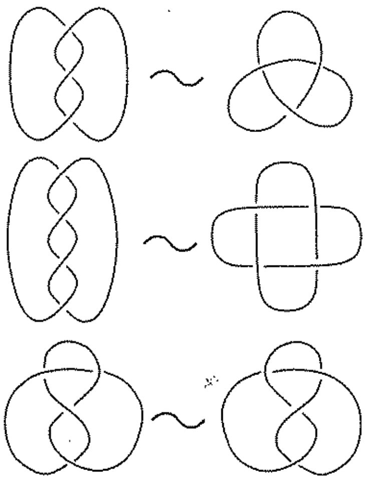
- Chirality of spatial graphs (Satoh)



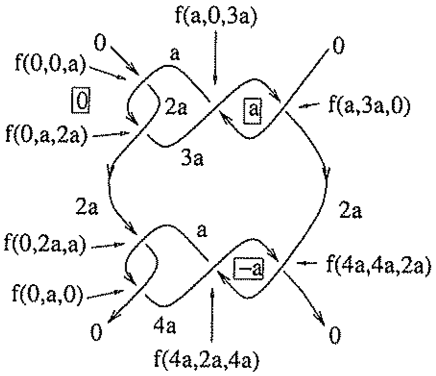
- Colored chirality



- Minimal number of type III Reidemeister moves



- Tangle embeddings



- Extension of colorings

- Chirality of knots and graphs

Lie Theory

A **Lie group** is a group G that is also a manifold where the maps

$$m \colon G \times G \to G \text{ and } inv \colon G \to G$$

defined by

$$m(a,b) = ab \text{ and } inv(a) = a^{-1}$$

are smooth.

A **Lie algebra** is a vector space L with an operation called the **bracket**

$$[\cdot,\cdot] \colon L \times L \to L$$

that is bilinear, skew-symmetric, and satisfies the **Jacobi identity**

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

for all $x, y, z \in L$.

Fact: Let G be a Lie group. The tangent space T_eG of G at the identity e is a Lie algebra. We say it is **the Lie algebra of** G and write $\mathfrak{g}$.

Quandles and Lie Theory

- Negatives in Lie algebra correspond to inverses in Lie group
- Addition in Lie algebra corresponds to multiplication in Lie group

Puzzle: What does the bracket correspond to?

Answer: *Conjugation!*

Moreover, both quandles and Lie algebras give a solution of the Yang–Baxter equation!

Proposition. Let L be a vector space over k equipped with a skew-symmetric bilinear operation

$$[\cdot,\cdot]\colon L \times L \to L.$$

Let $L' = k \oplus L$ and define the isomorphism

$$B\colon L' \otimes L' \to L' \otimes L' \text{ by}$$

$$B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y]).$$

Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

In fact, this space L' resembles a quandle! Define:

$$(a,x) \lhd (b,y) = (ab, bx + [x,y])$$

This satisfies the self-distributive law of a quandle!

Internalization

Familiar mathematical concepts live in the world of sets, but many can live in other worlds as well! We call this **internalization**.

How do we do this?

- Given some concept, express its definition completely in terms of commutative diagrams.
- Now interpret these diagrams in some ambient category K.

We actually do this all the time! Topological groups, Lie groups, etc.

Lie algebras from Lie groups via Quandles

$$\begin{tikzcd}
\text{Lie groups} \arrow[d] & \\
\text{Groups in Diff}_* \arrow[r, "U"] \arrow[d] & \text{Diff}_* \arrow[d, "J_\infty"] \\
\text{Groups in } \mathcal{C} \arrow[r, "U"] \arrow[d] & \mathcal{C} \arrow[d, "1"] \\
\text{Pointed Quandles in } \mathcal{C} \arrow[r, "U"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\text{Lie algebras} \arrow[r, "U"] & \text{Vect}
\end{tikzcd}$$

Internalized Quandles

Using the technique of internalization, we are able to talk about quandles living in the categories of commutative and cocommutative algebras!

- **Lie algebras**

$$(a, x) \lhd (b, y) = (ab, bx + [x, y])$$

- **Quandles/racks as bases** Let X be a rack. Let $V = kX$ be the vector space over a field k with the elements of X as basis. Set $W = k \oplus kX$. Define:

$$(a+\sum a_x x)\lhd(b+\sum b_y y) = \sum_y ab_y+\sum_{x,y} a_x b_y (x\lhd y)$$

- **Hopf algebras** Let H be a Hopf algebra. Define:

$$x \lhd y = S(y_{(1)}) x y_{(2)}$$

Questions

- What are more precise relationships among the Lie bracket, self-distributivity, solutions to the Yang-Baxter equations, Hopf algebras, and quantum groups?

- Can we define a cohomology theory for these internalized quandles that be used to construct invariants of knots and knotted surfaces?

- What's *really* going on here? What's the conceptual meaning behind all this?

- How can these observations be extended to higher dimensions, such as to higher dimensional Lie algebras, or Lie 2-algebras?

- How, if at all, do the Zamolodchikov tetrahedron equation and the Jacobiator identity of a Lie 2-algebra relate to this self-distributive story?