Musical Actions of Dihedral Groups

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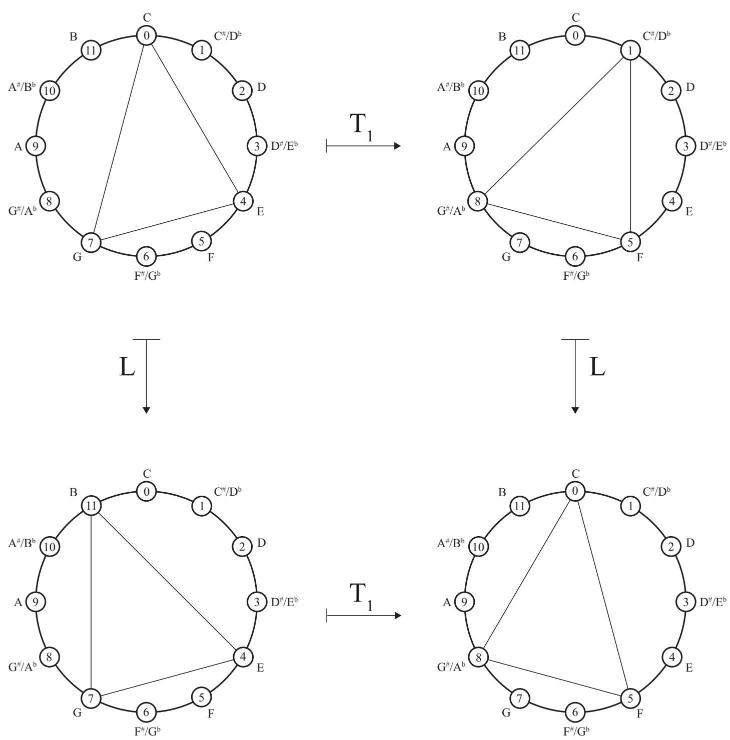

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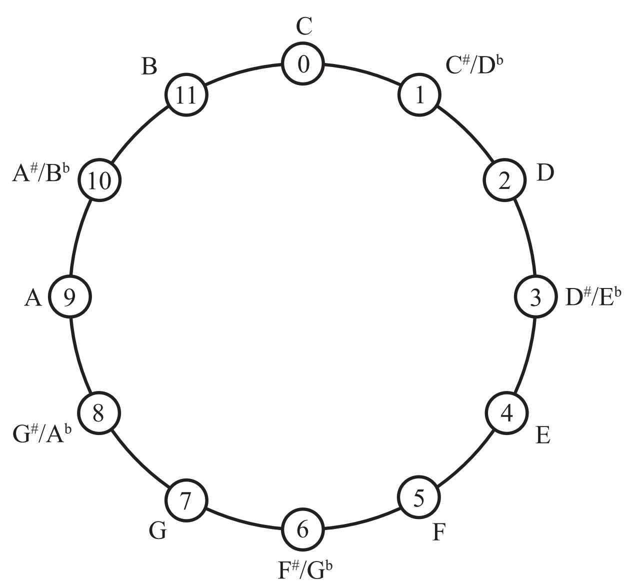
Pitch Classes and $\mathbb{Z}_{12}$

Two pitches are **enharmonically equivalent** if they are represented by the same key on a keyboard and thus are identical in pitch.

The interval between two consecutive pitches is called a **semitone**.

Musical Clock:



Transposition and Inversion

Mathematically, the functions of **transposition** and **inversion** are defined by:

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$
$$T_n(x) := x + n$$

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$
$$I_n(x) := -x + n$$

Proposition. (T/I group) The collection of transpositions and inversions form the dihedral group of order 24.

Major and Minor Triads

A **major triad** consists of a *root* notes, a second note 4 semitones above the root, and a third note 7 semitones above the root. A **minor triad** consist of a *root* note, a second note 3 semitones above the root, and a third note 7 semitones above the root.

Example. The C-major triad consists of $\{0, 4, 7\} = \{C, E, G\}$ while the f-minor triad consists of $\{5, 8, 0\} = \{F, A\flat, C\}$.

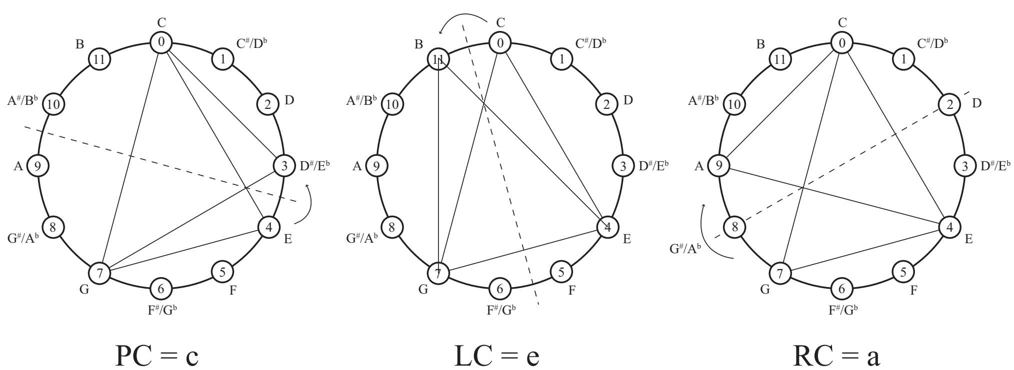
Major Triads	Minor Triads
$C = \langle 0, 4, 7 \rangle$	$\langle 0, 8, 5 \rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8 \rangle$	$\langle 1, 9, 6 \rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9 \rangle$	$\langle 2, 10, 7 \rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10 \rangle$	$\langle 3, 11, 8 \rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11 \rangle$	$\langle 4, 0, 9 \rangle = a$
$F = \langle 5, 9, 0 \rangle$	$\langle 5, 1, 10 \rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1 \rangle$	$\langle 6, 2, 11 \rangle = b$
$G = \langle 7, 11, 2 \rangle$	$\langle 7, 3, 0 \rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3 \rangle$	$\langle 8, 4, 1 \rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4 \rangle$	$\langle 9, 5, 2 \rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5 \rangle$	$\langle 10, 6, 3 \rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6 \rangle$	$\langle 11, 7, 4 \rangle = e$

The PLR Group

Let S be the set of major and minor triads.
Define $P, L, R : S \longrightarrow S$ by:

$$P\langle y_1, y_2, y_3\rangle = I_{y_1+y_3}\langle y_1, y_2, y_3\rangle$$
$$L\langle y_1, y_2, y_3\rangle = I_{y_2+y_3}\langle y_1, y_2, y_3\rangle$$
$$R\langle y_1, y_2, y_3\rangle = I_{y_1+y_2}\langle y_1, y_2, y_3\rangle$$

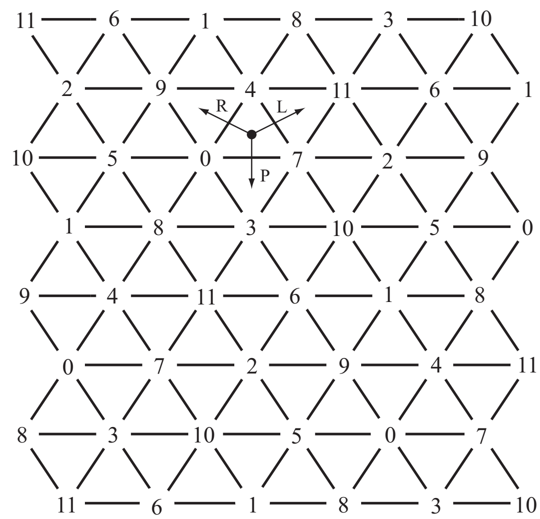
The PLR-**group** is the subgroup of the symmetric group on the set S generated by the bijections P, L, and R.



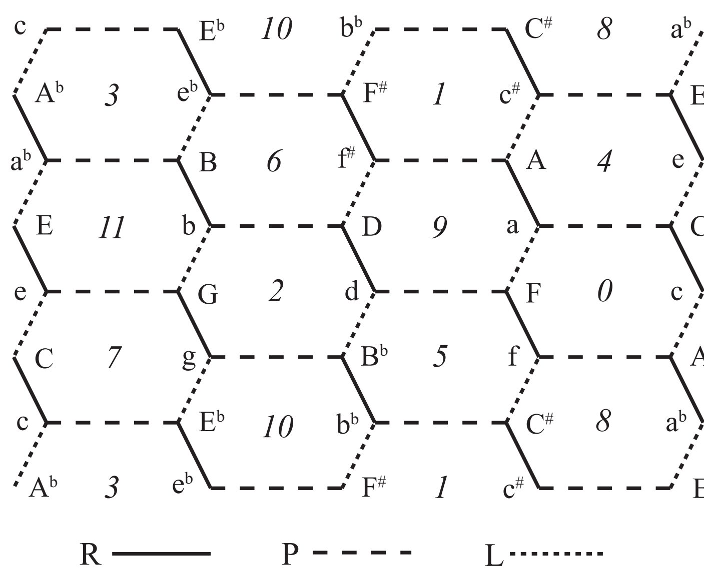
Theorem. The PLR-group is generated by L and R and is dihedral of order 24.

Geometric Description of PLR

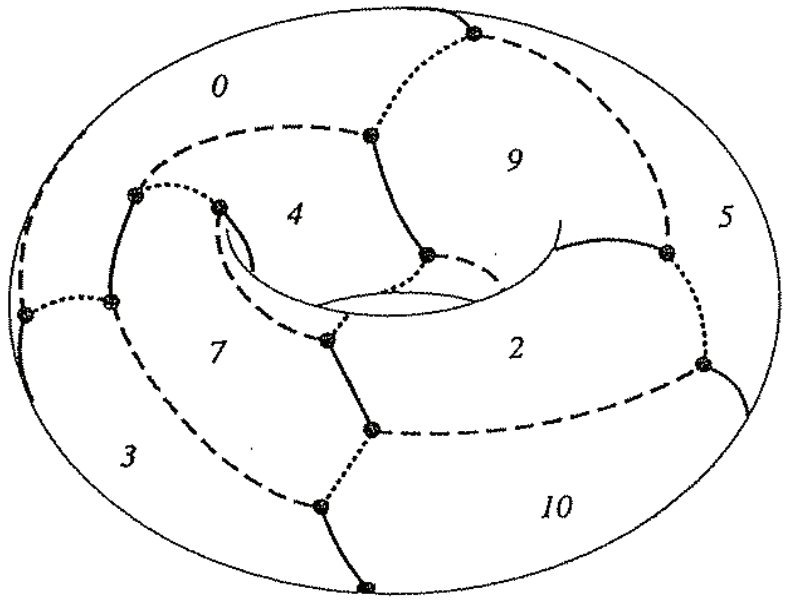
The Oettingen/Riemann *Tonnetz*:



Douthett and Steinbach:



Chicken Wire Torus:

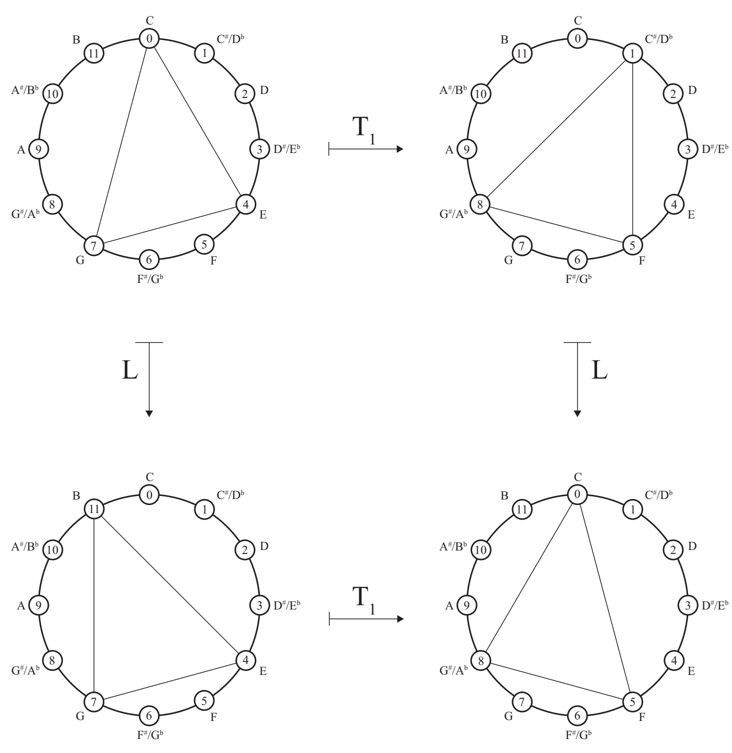


Duality!

We have seen that the dihedral group of order 24 acts on the set of major and minor triads:

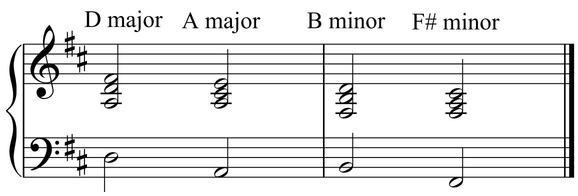
- through the T/I group
- through the PLR group

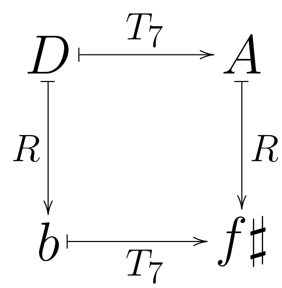
Proposition. The T/I group and PLR group are *dual*!



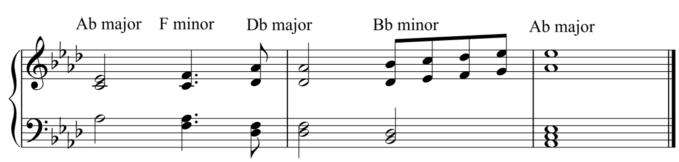
Musical Examples

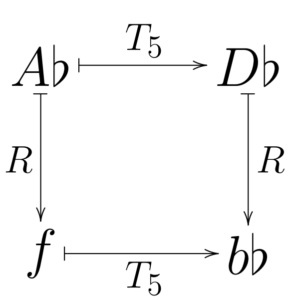
- Pachelbel's *Canon in D*



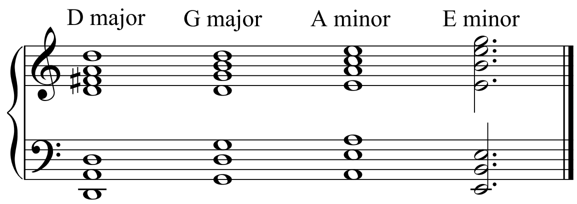


- Wagner's "Grail" theme from *Parsifal*





- Ives' *Religion*



$$\begin{array}{ccc} D & \xmapsto{I_6} & a \\ \Big\downarrow LR & & \Big\downarrow LR \\ G & \xmapsto[I_6]{} & e \end{array}$$