Braids and the Yang-Baxter Equation:

An application of knot theory to physics

Alissa S. Crans
University of California, Riverside
E-mail: acrans@math.ucr.edu
January, 2004

Gauge theories

- Electromagnetism

- Strong Force–holds quarks together in particles like protons and neutrons

- Weak Force–lets one sort of quark or lepton turn into another, which is the process responsible for many radioactive decays in nuclei and also occurs in nuclear fusion

- Gravity

Definitions

- A **group** G is a set equipped with a binary operation $\cdot : G \times G \to G$, called *multiplication*, an operation $^{-1} : G \to G$, called the *inverse* map, and a special element $1 \in G$, called the *identity*, such that for all $g, h, k \in G$ we have:

  1. $(g \cdot h) \cdot k = g \cdot (h \cdot k)$
  2. $g \cdot 1 = 1 \cdot g = g$
  3. $g \cdot g^{-1} = g^{-1} \cdot g = 1$

- We say a group G is a **Lie group** if it is a manifold and the product and inverse operations are smooth maps.

Examples of Lie groups

1. **General Linear Group:**
   $GL(n)$= {invertible Real $n \times n$ matrices}

2. **Special Linear Group:**
   $SL(n)$ = {Real $n \times n$ matrices with determinant 1}

3. **Orthogonal Group:**
   $O(n)$ = {orthogonal Real $n \times n$ matrices}

4. **Special Orthogonal Group:**
   $SO(n)$ = {orthogonal Real $n \times n$ matrices with determinant 1}

5. **Unitary Group:**
   $U(n)$ = {unitary Complex $n \times n$ matrices}

6. **Special Unitary Group:**
   $SU(n)$ = {unitary Complex $n \times n$ matrices with determinant 1}

Beautiful Fact

Different Lie groups give different equations called *Yang-Mills equations*, which describe various forces in the Standard Model!

The group is called the **symmetry** or **gauge** group of the force in question.

- Electromagnetism–$U(1)$
- Strong Force–$SU(3)$
- Electroweak Force–$SU(2) \times U(1)$
- Internal symmetry group of Standard Model–$SU(3) \times SU(2) \times U(1)$

Definitions

- Let G be a Lie group. We define the **Lie algebra** of G, written $\mathfrak{g}$, to be the tangent space at the identity element of G.

- A **Lie algebra** is a vector space $\mathfrak{g}$ equipped with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for all $x, y, z \in \mathfrak{g}$ and scalars α, β we have:

  1. $[x, y] = -[y, x]$
  2. $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]$
  3. *Jacobi identity*
     $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Examples of Lie algebras

1. $\mathfrak{gl}(n) = \{$Real $n \times n$ matrices$\}$

2. $\mathfrak{sl}(n) = \{$traceless Real $n \times n\}$

3. $\mathfrak{o}(n) = \{$skew-adjoint Real $n \times n$ matrices$\}$

4. $\mathfrak{so}(n) = \{$traceless, skew-adjoint Real $n \times n$ matrices$\}$

5. $\mathfrak{u}(n) = \{$skew-adjoint Complex $n \times n$ matrices$\}$

6. $\mathfrak{su}(n) = \{$traceless, skew-adjoint Complex $n \times n$ matrices$\}$

Interesting Observation!

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Given $x, y \in \mathfrak{g}$, and $s, t \in \mathbb{R}$, we can form $e^{sx}, e^{ty} \in G$.

$$\frac{d}{ds}\frac{d}{dt}e^{sx}e^{ty}e^{-sx}|_{s,t=0} = [x, y]$$

Conjugation

No more than these three axioms are necessary to capture the equational content of conjugation:

1. $ggg^{-1} = g$

2. $g^{-1}(ghg^{-1})g = h = g(g^{-1}hg)g^{-1}$

3. $g(hkh^{-1})g^{-1} = (ghg^{-1})(gkg^{-1})(ghg^{-1})^{-1}$

Writing

$$\begin{aligned} ghg^{-1} &= g \triangleright h \\ g^{-1}hg &= h \triangleleft g \end{aligned}$$

we obtain...

Further Fun!

- *Linking number* and electromagnetism

- *Knot invariants* and physics

  1. Jones polynomial and spin-$\frac{1}{2}$ reps of $SU(2)$
  2. HOMFLY polynomial and fundamental rep of $SU(n)$
  3. Kauffman polynomial and fundamental rep of $SO(n)$

- *Knot theory* and 2-dimensional statistical mechanics

- *Kauffman bracket* and Chern-Simons theory

- Quandles and Knot crystals

Texts

- *Gauge Fields, Knots and Gravity*, J. Baez & J. Muniain

- *Differential Forms in Algebraic Topology*, R. Bott & L. Tu

- *Knots and Physics*, Louis Kauffman