2-Quandles: Categorified Quandles

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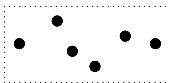
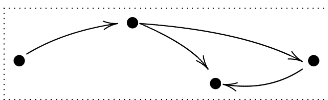
1. **Project Questions**

One of the most significant mathematical accomplishments of the early 20^{th} century was to formalize all of mathematics in terms of the language of set theory. The language of Zermelo-Fraenkel set theory, the most popular of a number of successful approaches, is based on two fundamental relationships between sets: membership and equality. That is, two sets are equal if and only if they have the same elements, or members. Set theory plays a central role in mathematics because it provides a basic language in which most of mathematics is expressed. Indeed, most mathematical structures amount to a set equipped with extra bells and whistles. The deficiency of this language, however, is that it regards *things* as more fundamental than *processes* or *relationships*. Thus, set theory is often inadequate when describing problems in physics and computer science, where the process is often just as, if not more, important than the things in question.

More recently, in an attempt to give processes and relationships an equal status with things, Eilenberg and MacLane developed a more flexible language: the language of category theory. This language was introduced in the 1940's as a tool for illuminating the relationship between algebra and topology. In category theory, the role of set is replaced by the notion of a *category*. Roughly speaking, a **category** consists of a collection of **objects** (analogous to the elements of a set) and for each pair of objects, a collection of **morphisms** from one object to another. In other words, we have things and ways to go between those things. In this light, a category can be regarded as the higher-dimensional analog of a set. While we can visualize a set as just a bunch of 0-dimensional points corresponding to its elements:

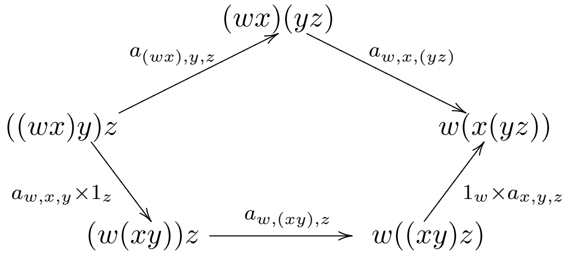


we can visualize a category as a bunch of points corresponding to its objects, together with 1-dimensional arrows corresponding to its morphisms:



Just as we have functions between sets, we have *functors* between categories, and just as we have equations between functions, we have *natural transformations* between functors. Roughly speaking, a **functor** is a morphism between categories that preserves all structure in sight. A **natural transformation** is a morphism *between* functors. That is, it is a morphism between morphisms between categories. Due to its emphasis on morphisms, the language of category theory advocates a way of thinking in which we describe things by their relationships to other things rather than in terms of their parts.

In the mid-1990's, Crane [C, CF] coined the term *categorification* to refer to the process of developing category-theoretic analogs of set-theoretic concepts. In this process we replace sets with categories, functions with functors, and equations between functions by natural isomorphisms between functors. We then require that these isomorphisms satisfy equations of their own, called coherence laws. Finding the correct coherence laws is often the most difficult aspect of this generalization process. For example, the category of finite sets together with functions as morphisms is a categorification of the set of counting numbers. The functions of sum and product are replaced by the functors disjoint union and Cartesian product. The equational laws satisfied by addition and multiplication such as commutativity, associativity, and distributivity, now hold for disjoint union and Cartesian product, but only up to natural isomorphism. For instance, the associative law, $(xy)z = x(yz)$, is replaced by a natural isomorphism: $a_{x,y,z} : (xy)z \xrightarrow{\sim} x(yz)$, which we then require to satisfy a coherence law known as the pentagon identity:



Perhaps the greatest strength of categorification is that it allows us to refine our notion of 'sameness' by enabling us to distinguish between equality and isomorphism. In a set, two elements are either the same or different, while in a category, two objects can be 'the same in a way' while remaining different. That is, they can be isomorphic, but not equal. Better still, we are able to explicitly keep track of *how* two objects are the same: the isomorphism itself. Moreover, two objects can be the same in multiple ways, since there can be various different isomorphisms between them. This more careful consideration of the notion of sameness is the reason that categorification plays an increasingly important role not only in mathematics, but also in physics and computer science, where symmetry plays a significant role and a precise treatment of the notion of sameness is crucial.

My goal for this project is to obtain the definition of a *2-quandle*, which will be the categorified version of an algebraic structure known as a *quandle*. A **quandle** is a set Q equipped with two binary operations satisfying axioms that capture the essential properties of the operations of conjugation in a group and algebraically encode the three Reidemeister moves. In 1982, Joyce [J] introduced quandles as a source knot invariants, and they have been studied and explored since then in various papers [Bri, FR, K].

I have numerous reasons for this project choice. My research thus far has consisted of an investigation of Lie 2-groups and Lie 2-algebras, the categorified versions of Lie groups and Lie algebras. I hope to further my previous work through this proposed project by defining 2-quandles and relating them to Lie 2-groups and Lie 2-algebras. I suspect that just as quandles provide knot invariants, 2-quandles will provide invariants of the higher-dimensional analogs of traditional knots and braids, called *2-knots* and *2-braids*, introduced by Carter and Saito [CS] in the late 1990's. Furthermore, I hope to classify 2-quandles using quandle cohomology [CJKLS] just as I was able to classify Lie 2-algebras using Lie algebra cohomology in my dissertation. If time permits, I then plan to use 2-quandles to illustrate the passage from a Lie 2-group to its Lie 2-algebra, just as quandles provide a conceptual explanation of the passage from a Lie group to its Lie algebra.

2. Literature Review

In the early 1990's, Crane [C, CF] formalized the process of categorification. More recently, Baez and Dolan [BD] summarized Crane's work while illuminating its relationship to topology. With the exception of my work [BC], only the notions of categorified groups [BL] and Hilbert spaces [B] have been defined and explored significantly.

As previously mentioned, quandles were defined by the topologist David Joyce [J] in the early 1980's and further analyzed since then by others [Bri, FR, K]. In 2001, Carter, Jelsovsky, Kamada, Langford, and Saito [CJKLS] described the definition of quandle cohomology, with the goal of obtaining knot invariants. In the late 1990's, the higher-dimensional analogs of knots and braids, known as 2-knots and 2-braids, originated in the work of Carter and Saito [CS]. To date, very little other than the definitions of these notions is known. My goal is to relate these ideas.

3. Method

As mentioned in Section 1, the most difficult aspect of this process will consist of finding the correct coherence laws that a 2-quandle must satisfy. Fortunately, since I seek to demonstrate a connection between 2-quandles and the 2-braids of Carter and Saito [CS], I can take a cue from topology to assist in this categorification. I intend to examine the basic laws satisfied by 2-braids and then adapt them accordingly to formulate a working definition of a 2-quandle. I will know that my definition is correct when 2-quandles provide invariants of these 2-braids. If this result does not hold, then I will continue to alter my proposed definition until it does. Once I have a definition satisfying this requirement, I will attempt to classify 2-quandles using quandle cohomology in a method analogous to the classification of Lie 2-algebras found in my dissertation. If time permits, I will categorify the passage from a Lie group to its Lie algebra via quandles, which I described in my dissertation, to explain the passage from a Lie 2-group to its Lie 2-algebra.

4. Timeline of Project

I have already begun thinking about this project and will continue to do so throughout the year. During the year, I will meet weekly with Sam Nelson, an expert in quandles, to ensure that I capture the essential qualities of quandles while categorifying. By the end of the spring semester 2005, I hope to already have a working definition of a 2-quandle which I can then test by examining whether these proposed 2-quandles provide invariants of 2-braids. During the summer of 2005, once I have a correct definition, I will then focus on the classification of 2-quandles as well as their relationship to Lie 2-algebras and Lie 2-groups. At the end of the summer I plan to organize my results into a publishable form.

5. Plan for Dissemination

My plans for dissemination include both publishing my results in a refereed mathematical journal such as *Topology*, *Geometry and Topology*, *Journal of Knot Theory and its Ramifications*, or *Theory and Applications of Categories*, as well as presenting my work in local seminars, such as the Claremont College colloquium and UCSD topology seminar. Moreover, I plan to present my work at the national meetings of the American Mathematical Society in January 2006 and at 'Groupoidfest 2005', which is a annual conference devoted to category theory. Finally, I intend to seek future support from the National Science Foundation (NSF) to further this work with 2-quandles and their relationships to other objects.

6. Significance of the work

The ability to distinguish knots and braids in 3 dimensions plays a prominent role in topology. Since we know little other than the definitions of knots and braids in 4 dimensions (that is, 2-knots and 2-braids), 2-quandles could provide great insight into understanding these objects from an algebraic viewpoint. Moreover, the problem of finding the Lie algebra of a given Lie group holds a very prominent place in the field of differential geometry. The analogous question for Lie 2-algebras and Lie 2-groups, then, should also play an important role in the field of categorified Lie theory, with 2-quandles supplying the machinery to perform this passage.

References

[B] J. Baez, Higher-dimensional algebra II: 2-Hilbert spaces, *Adv. Math.* **127** (1997), 125–189.

[BC] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, to appear in *Theory and Applications of Categories*, available as math.QA/0307263

[BD] J. Baez and J. Dolan, Categorification, in *Higher Category Theory*, eds. E. Getzler and M. Kapranov, Contemporary Mathematics vol. 230, AMS, Providence, 1998. Also available as math.QA/9802029.

[BL] J. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups, available as math.QA/0307200

[Bri] E. Brieskorn, Automorphic sets and singularities, Contemporary Math. 79 (1998), 45-115. MR **90a:**32024

[CS] J. S. Carter and M. Saito, *Knotted Surfaces and Their Diagrams*, American Mathematical Society, Providence, 1998.

[CJKLS] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, Quandle Cohomology and state-sum Invariants of Knotted Curves and Surfaces, available as math.GT/9903135.

[C] L. Crane, Clock and category: is quantum gravity algebraic?, *Jour. Math. Phys.* **36** (1995), 6180–6193.

[CF] L. Crane and I. Frenkel, Four dimensional topological quantum field theory, Hopf categories, and the canonical bases, *Jour. Math. Phys.* **35** (1994), 5136–5154.

[EG] P. Etingof and M. Grana, On rack cohomology, available as math.QA/0201290.

[FR] R. Fenn, C. Rourke, Racks and Links in codimension two, Journal of Knot Theory and its Ramifications 1 (1992), no. 4, 343-406. MR **94e:**57006

[J] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Alg., 23, (1982), 37-65.

[K] L.H. Kauffman, Knots and physics, World Scientific, Series on knots and everythihg, vol. 1, 1991. MR**93b:**57010