Research Summary

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Introduction

Higher-dimensional algebra is the study of generalizations of algebraic concepts obtained through a process called 'categorification'. My past research in higher-dimensional algebra consisted of developing and exploring categorified Lie algebras, also called Lie 2-algebras, and their relationships to low-dimensional topology. My current interests include developing the theory of categorified 'quandles.'

In the mid-1990's, Crane [C, CF] coined the term categorification to refer to the process of developing category-theoretic analogs of set-theoretic concepts. In this process we replace elements with objects, sets with categories, and functions with functors. We replace equations between elements by isomorphisms between objects, and replace equations between functions by natural isomorphisms between functors. Finally, we require that these isomorphisms satisfy equations of their own, called coherence laws. Finding the correct coherence laws is often the most difficult aspect of this generalization process. Ultimately, by iterating this process, mathematicians wish to obtain and apply the n-categorical generalizations of as many mathematical concepts as possible to strengthen and simplify the connections between different subfields of mathematics.

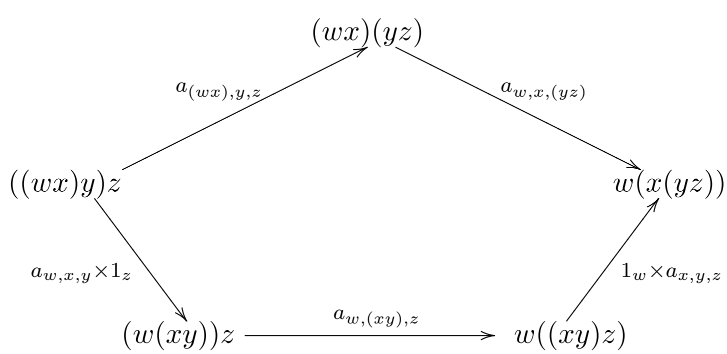
For example, the category of finite sets together with functions is a categorification of the set of natural numbers. The functions of sum and product in $\mathbb{N}$ are replaced by the functors disjoint union and Cartesian product of finite sets. The equational laws satisfied by addition and multiplication in $\mathbb{N}$, such as commutativity, associativity, and distributivity, now hold for disjoint union and Cartesian product, but only up to natural isomorphism. For instance, the associative law

$$(xy)z = x(yz)$$

is replaced by a natural isomorphism called the *associator*:

$$a_{x,y,z} \colon (xy)z \xrightarrow{\sim} x(yz)\,,$$

which we then require to satisfy a coherence law known as the pentagon identity:



Perhaps the greatest strength of categorification is that it allows us to refine our concept of 'sameness' by enabling us to distinguish between equality and isomorphism. In a set, two elements are either the same or different, while in a category, two objects can be isomorphic, but not equal. This more careful consideration of the notion of sameness is the reason that categorification plays an increasingly important role not only in mathematics, but also in physics and computer science, where a precise treatment of the notion of sameness is crucial.

Lie 2-Algebras

A Lie 2-algebra blends the notion of a Lie algebra with that of a category. Just as a Lie algebra has an underlying vector space, a Lie 2-algebra has an underlying 2-vector space. A 2-vector space is a hybrid of the notions of vector space and category. That is, it is a category where everything is *linear*. More precisely, a **2-vector space** V is a category consisting of vector spaces $\mathrm{Ob}(V)$ and $\mathrm{Mor}(V)$ of objects and morphisms, respectively, together with linear source and target maps $s, t\colon \mathrm{Mor}(V) \to \mathrm{Ob}(V)$, a linear identity-assigning map $i\colon \mathrm{Ob}(V) \to \mathrm{Mor}(V)$, and a linear composition map $\circ\colon \mathrm{Mor}(V) \times_{\mathrm{Ob}(V)} \mathrm{Mor}(V) \to \mathrm{Mor}(V)$. My paper with John Baez [BC] contains a development of the theory of 2-vector spaces. This new theory of 2-vector spaces has already begun to play a role in the representation theory of categorified groups, or 2-groups, as well as in topological quantum field theory [E,Ga,P].

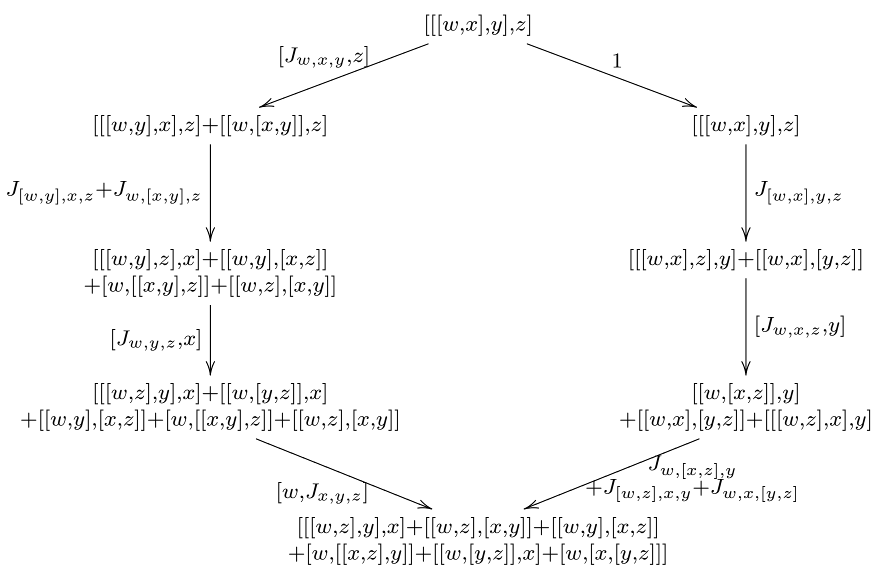
To obtain a **Lie 2-algebra**, we begin with a 2-vector space L and equip it with a bilinear, skew-symmetric bracket *functor*

$$[\cdot,\cdot]\colon L \times L \to L,$$

which satisfies the Jacobi identity *up to a natural isomorphism* called the 'Jacobiator',

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

The Jacobiator is completely antisymmetric and trilinear and satisfies a law called the 'Jacobiator identity', which says which says the octagon below commutes



for all $w, x, y, z \in \mathrm{Ob}(L)$. This octagon expresses that the two ways of using the Jacobiator to rebracket the expression $[[[w,x],y],z]$ are the same.

Relation to Lie Algebra Cohomology

Since the correct notion of sameness for categories is equivalence rather than isomorphism, the same is true for Lie 2-algebras. In my dissertation, I classified Lie 2-algebras up to equivalence in terms of third cohomology classes in Lie algebra cohomology:

Theorem.[Cr] *There is a one-to-one correspondence between equivalence classes of Lie 2-algebras and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space V, a representation ρ of $\mathfrak{g}$ on V, and an element of $H^3(\mathfrak{g}, V)$.*

This result provides a new interpretation of $H^3(\mathfrak{g}, V)$ in terms of Lie 2-algebras.

One of the more interesting examples of a finite-dimensional Lie 2-algebra, characterized in terms of the quadruple described above, consists of: a finite-dimensional Lie algebra $\mathfrak{g}$ over the field k, a vector space k, the trivial representation ρ of $\mathfrak{g}$ on k, and the 3-cocycle $\langle x, [y, z]\rangle$ where $\langle \cdot, \cdot \rangle$ is the Killing form.

In fact, every finite-dimensional Lie algebra $\mathfrak{g}$ admits a one-parameter deformation $\mathfrak{g}_\hbar$ in the category of Lie 2-algebras by choosing the 3-cocycle $\hbar\langle x, [y, z]\rangle$ where $\hbar$ is any element of k. Another construction of $\mathfrak{g}_\hbar$ has recently appeared in the literature [W] and part of my future work includes demonstrating that this version is equivalent to the one in my dissertation. I suspect that such Lie 2-algebras have connections to quantum groups and affine Lie algebras and intend to pursue this relationship further. Moreover, I desire to explore the representation theory of Lie 2-algebras beginning with the Lie 2-algebras $\mathfrak{g}_\hbar$.

Relation to Lie 2-Groups

Since theories of 2-groups and Lie 2-groups already exist [BLau], a natural question is whether Lie 2-algebras arise from Lie 2-groups. As a result of a collaborative effort, it has been shown that we can construct for integral values of $\hbar$ an infinite-dimensional Lie 2-group whose Lie 2-algebra is *equivalent* to $\mathfrak{g}_\hbar$:

Theorem.[BCSS] *Let G be a simply-connected compact simple Lie group. For any $\hbar \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_\hbar G$ whose Lie 2-algebra $\mathcal{P}_\hbar \mathfrak{g}$ is equivalent to $\mathfrak{g}_\hbar$.*

Recently, this result has been complemented by work of Getzler and Henriques [G, H]. The result in [BCSS] is of special interest because it provides an interesting relationship between Lie 2-algebras, the Kac–Moody central extensions of loop groups, and the group String(n), which plays a prominent role in string theory:

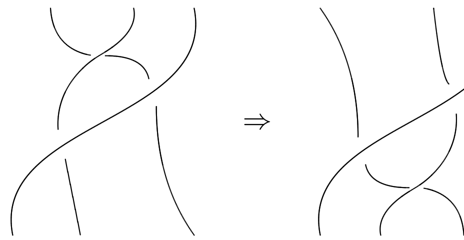
Theorem.[BCSS] *Let G be a simply-connected compact simple Lie group. Then $|\mathcal{P}_\hbar G|$ is an extension of G by a topological group that is homotopy equivalent to $K(\mathbb{Z}, 2)$. Moreover, $|\mathcal{P}_\hbar G| \simeq \hat{G}$ when $\hbar = \pm 1$.*

This theorem provides a new construction of one model of the group String(n) that is wholly different from those of Stolz and Teichner.

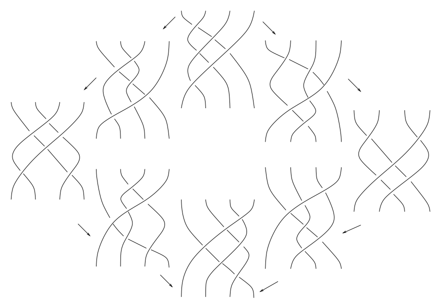
Relation to Topology

The coherence law for the Jacobiator is intimately related to the Zamolodchikov tetrahedron equation. This equation plays a role in the theory of knotted surfaces in 4-space analogous to that played by the Yang–Baxter equation, or third Reidemeister move, in the theory of knots in 3-space. Since the algebraic version of this equation is not particularly illuminating, we offer a geometric description instead.

Consider the surface in 4-space traced out by the *process of performing* the third Reidemeister move:



The Zamolodchikov tetrahedron equation says the surface traced out by first performing the third Reidemeister move on a threefold crossing and then sliding the result under a fourth strand is isotopic to that traced out by first sliding the threefold crossing under the fourth strand and then performing the third Reidemeister move. The following commutative octagon formalizes this process:



Any Lie algebra gives a solution of the Yang–Baxter equation. In fact, under suitable conditions, the Yang–Baxter equation and Jacobi identity are actually equivalent. My dissertation contains a higher-dimensional analog:

Theorem.[Cr] *Any Lie* 2*-algebra gives a solution of the Zamolodchikov tetrahedron equation.*

That is, under suitable conditions, the Zamolodchikov tetrahedron equation is equivalent to the Jacobiator identity.

Recent Work

Physicists and mathematicians alike are interested in universal solutions to the Yang–Baxter equation coming from various algebraic structures such as vector spaces, quantum groups, and the like. Such interest in these solutions exists since they have applications in the study and classification of knotted loops in space. In my work with Baez and Wise [BCW], we illustrated connections between these ideas and elements of string theory. In particular we show that loop-like defects in 4d BF theory obey exotic statistics governed by the 'loop braid group,' which turns out to be isomorphic to the 'braid permutation group' of Fenn, Rimányi and Rourke. We also discuss 'quandle field theory', in which the gauge group G is replaced by an algebraic structure called a 'quandle'. Roughly, a *quandle* is a set equipped with two binary operations satisfying axioms that capture the essential properties of group conjugation and algebraically encode the three Reidemeister moves from knot theory.

The intimate relationship between Lie theory and braid theory evidenced by the results given in the previous section inspired a desire to further investigate the characteristics of algebraic structures that provide solutions of the Yang–Baxter equation, which led to to an ongoing collaboration with Carter, Elhamdadi, and Saito. The major themes of our recent works include using quandle cocycles to develop non-trivial invariants of classical knots and knotted surfaces [CJKLS]. The quandle homology theory has been generalized to a theory encompassing set-theoretic solutions to the Yang-Baxter equations [CES], and to similar homology theories for Frobenius algebras [CCEKS] and the adjoint map of Hopf algebras [CCES2]. Moreover, quandle cocycles and Lie algebra cocycles fit into a single diagrammatically defined cohomology of self-distributive maps in a cocommutative coalgebra [CCES1] which can be used to study deformations of these self-distributive structures.

Future Work

We have numerous goals for our continued work, including relating the cocycles that appear in the self-distributive structures to Lie algebra cocycles since the former yield invariants of knots and we have evidence suggesting that the latter should as well. We desire to integrate quandle homology into mainstream homological algebra, using methodologies such as spectral sequences, the bar resolution, and quandles in crossed modules. Specifically, we seek answers to the following questions: What are more precise relationships among the Lie bracket, self-distributivity, solutions to the Yang-Baxter equations, Hopf algebras, and quantum groups? What are the relationships between the Frobenius cohomology theory, cohomology of the adjoint map in a Hopf algebra, self-distributive cohomology theory, Lie algebra cohomology, and quandle cohomology? We expect that there are connections among these cohomology theories that extend beyond their formal definitions and we anticipate topological, categorical, and/or physical applications because of the diagrammatic nature of these theories. Can the 3-cocycles from each of the new theories provide solutions to the Zamolodchikov equation? Can they be used to construct invariants of knots and knotted surfaces? How can the new theories be uniformly formulated? How can our results be extended to higher dimensions, such as to higher Lie theory, and in particular, to Lie 2-algebras? How do the Zamolodchikov tetrahedron equation and the Jacobiator identity relate to higher self-distributivity? Finally, how do we define higher-dimensional cocycles and will they provide invariants of knotted surfaces?

Related to this joint work, though not part of a collaborative effort, is my current work developing and exploring the theory of categorified quandles, or 2-quandles. I have already defined the notion of a strict 2-quandle and have obtained relationships between (strict) 2-quandles and 2-groups analogous to the relationships between quandles and groups. However, numerous questions remain, such as: Can this definition also be expressed in the language of crossed-modules, like the definitions of strict 2-groups and strict Lie 2-algebras? What is the definition of a weak 2-quandle? Can weak 2-quandles provide an explanation of the passage from Lie 2-groups to Lie 2-algebras, just as quandles provide a new, conceptual method of passing from Lie groups to Lie algebras [Cr]? Finally, can weak 2-quandles be classified up to equivalence in terms of third cohomology classes in quandle cohomology in analogy with the classification of Lie 2-algebras? These questions form the groundwork of my current and future study.

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