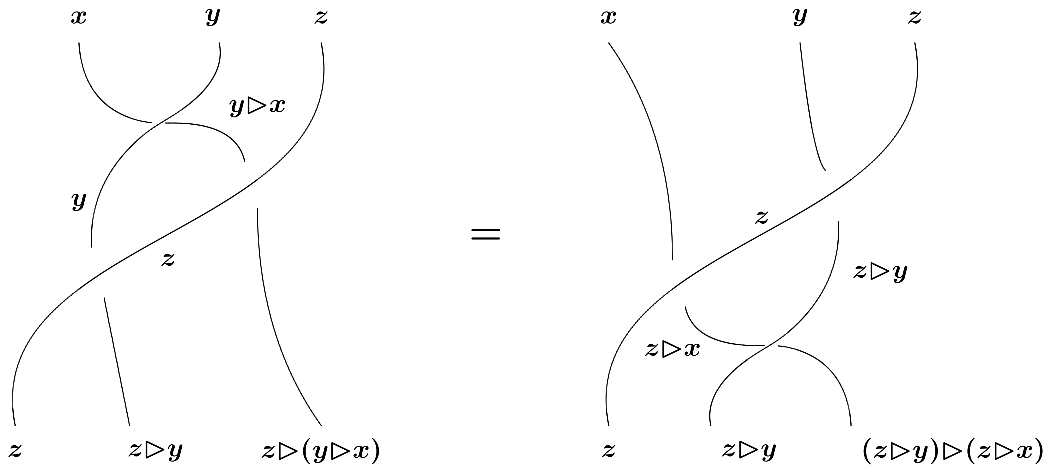
Categorical Self-Distributivity

Alissa S. Crans
Loyola Marymount University

Joint work with:

J. Scott Carter
Mohamed Elhamdadi
& Masahico Saito



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Self–Distributivity

Let X be a set equipped with a binary operation $\triangleleft : X \times X \to X$. The map $\triangleleft$ satisfies the **self-distributive law** if

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

Examples:

- Let G be a group and

  $$a \triangleleft b = b^{-1}ab$$

  for $a, b \in G$

- Let G be a group and H a subgroup. Let s be an automorphism of G that fixes the elements of H. Then consider G/H with

  $$Ha \triangleleft Hb = Hs(ab^{-1})b$$

  for $a, b \in G$

- Let T be an invertible linear transformation on a vector space V. Consider V with

  $$x \lhd y = T^{-1}(y - x) + y$$

  for $x, y \in V$

- Consider a $\mathbb{Z}([t, t^{-1}])$-module M with

  $$a \lhd b = ta + (1 - t)b$$

  for $a, b \in M$

- Consider $\mathbb{R}^2$ with

  $$b \lhd a = \text{“reflect b about a”} = 2a - b$$

  for $a, b \in \mathbb{R}^2$

- Consider $\mathbb{Z}_n$ with

  $$i \lhd j \equiv 2j - i \pmod{n}$$

  for $i, j \in \mathbb{Z}_n$

Yang–Baxter Equation

Given a set X and an isomorphism

$$B\colon X \times X \to X \times X,$$

we say B is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:

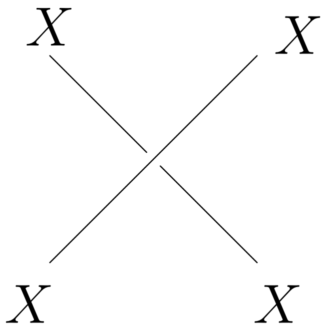
$$(B \times 1)(1 \times B)(B \times 1) = (1 \times B)(B \times 1)(1 \times B),$$

or in other words, that this diagram commutes:

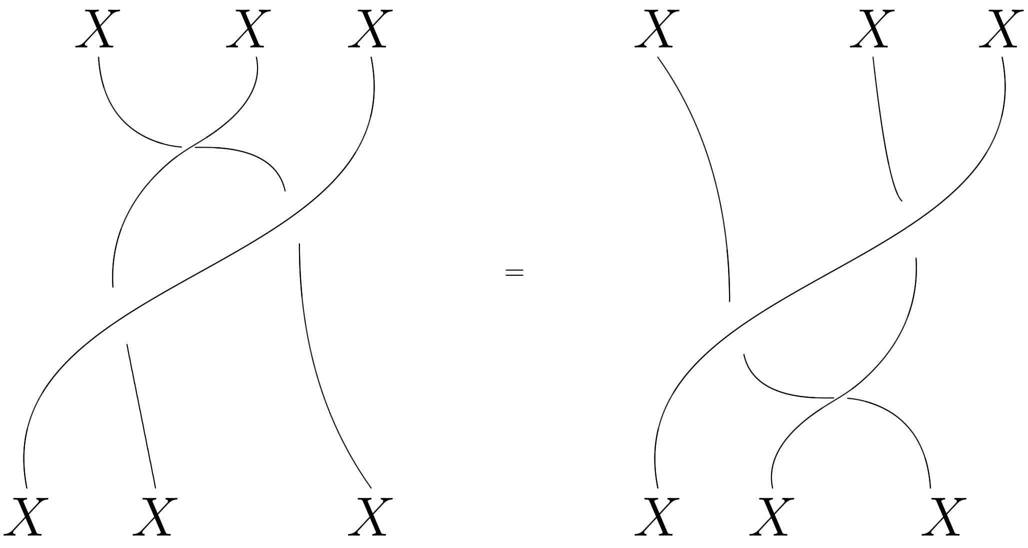
$$\begin{tikzcd}
 & X \times X \times X \arrow[dl, "1\times B"'] \arrow[dr, "B\times 1"] & \\
X \times X \times X \arrow[d, "B\times 1"'] & & X \times X \times X \arrow[d, "1\times B"] \\
X \times X \times X \arrow[dr, "1\times B"'] & & X \times X \times X \arrow[dl, "B\times 1"] \\
 & X \times X \times X &
\end{tikzcd}$$

If we draw $B\colon X \times X \to X \times X$ as a braiding:

B :



the Yang–Baxter equation says that:



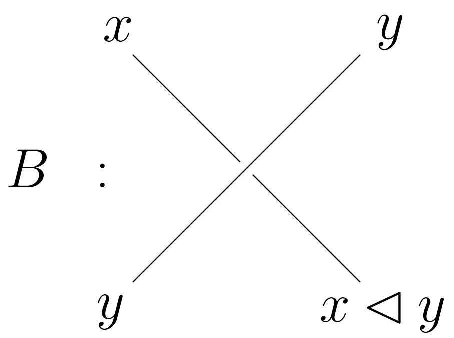
Solution of the Yang–Baxter Equation

Proposition. Let X be a set equipped with a binary operation $\triangleleft : X \times X \to X$ that satisfies the self-distributive law. We obtain a solution to the Yang–Baxter equation by defining

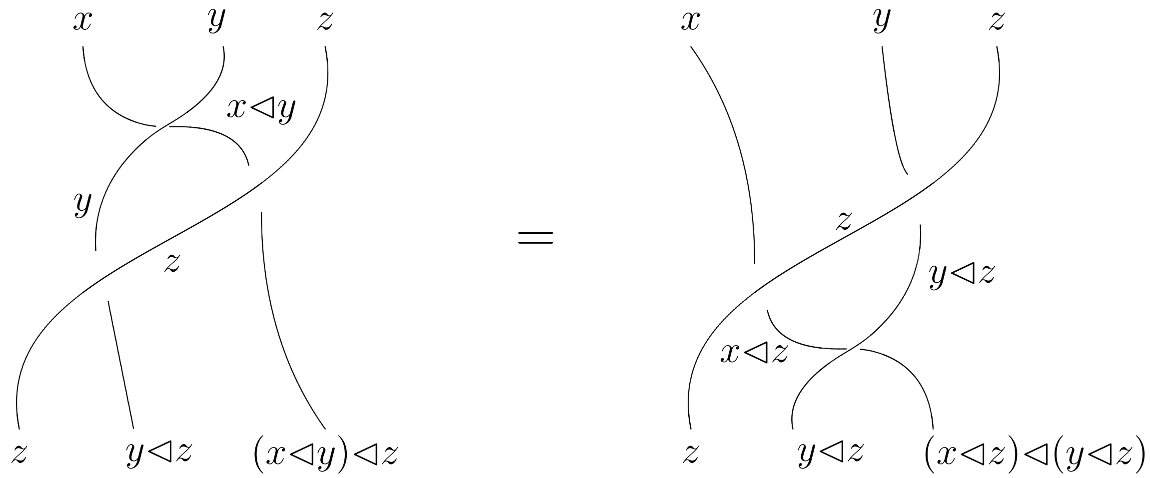
$$B : X \times X \to X \times X \;\text{ as}$$

$$B(x, y) = (y, x \triangleleft y)$$

Pictorially, B takes the form:



Proof.


x
y
z
$x\lhd y$
y
z
z
$y\lhd z$
$(x\lhd y)\lhd z$
$=$
x
y
z
z
$y\lhd z$
$x\lhd z$
z
$y\lhd z$
$(x\lhd z)\lhd(y\lhd z)$

Motivating Example

A **Lie algebra** is a vector space L over k equipped with an operation called the **bracket**

$$[\cdot, \cdot]\colon L \times L \to L$$

that is bilinear, skew-symmetric, and satisfies the **Jacobi identity**

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

for all $x, y, z \in L$.

Examples:

- $\mathbb{R}^3$ with $[x, y] = x \times y$ (cross product)
- $\mathfrak{gl}(n)$ = Real $n \times n$ matrices with $[A, B] = AB - BA$
- $\mathfrak{sl}(n)$ = traceless Real $n \times n$ with $[A, B] = AB - BA$

Proposition. Let L be a vector space over k equipped with a skew-symmetric bilinear operation

$$[\cdot,\cdot]\colon L \times L \to L$$

Let $L' = k \oplus L$ and define an isomorphism

$$B\colon L' \otimes L' \to L' \otimes L' \text{ by}$$

$$B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y]).$$

Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

Goals

- Formulate an analogue of self-distributivity for coalgebras and Hopf algebras using comultiplication

- Propose a cohomology theory that encompasses both Lie algebra and quandle cohomologies, is analogous to Hochschild cohomology, and can be used to study deformations of these self-distributive structures

Generalized Self-Distributivity

Let X be an object in a category K with finite products. A map $q : X \times X \to X$ **satisfies the self-distributive law** if

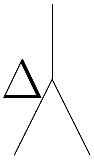
$$q(q \times 1) = q(q \times q)(1 \times \tau \times 1)(1 \times 1 \times \Delta),$$

or in other words, that this diagram commutes:

$$\begin{tikzcd}
 & & X \times X \times X \arrow[dll, "1\times1\times\Delta"'] \arrow[drr, "q\times1"] & & \\
X \times X \times X \times X \arrow[d, "1\times\tau\times1"'] & & & & X \times X \arrow[d, "q"] \\
X \times X \times X \times X \arrow[dr, "1\times1\times q"'] & & & & X \\
 & X \times X \times X \arrow[rr, "q\times1"'] & & X \times X \arrow[ur, "q"'] &
\end{tikzcd}$$

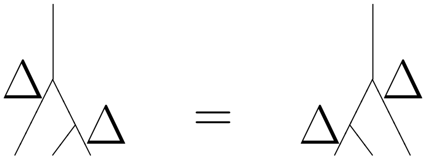
where $\Delta : X \to X \times X$ is the diagonal morphism in K and $\tau : X \times X \to X \times X$ is the transposition.

A **coalgebra** is a vector space C equipped with a **comultiplication** map $\Delta\colon C \to C \otimes C$



that is bilinear and **coassociative**:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

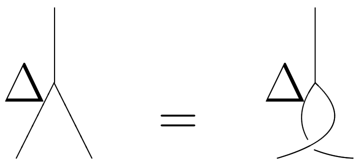


A **coalgebra with counit** is a coalgebra with a **counit** $\epsilon\colon C \to k$ such that

$$\begin{tikzcd}
k\otimes C \arrow[r, "\epsilon\otimes 1"] \arrow[dr] & C\otimes C & C\otimes k \arrow[l, "1\otimes\epsilon"'] \arrow[dl] \\
& C \arrow[u, "\Delta"'] &
\end{tikzcd}$$

commutes.

A coalgebra is **cocommutative** if $\Delta = \tau\Delta$:



Lie Algebras and Self-Distributivity

Let L be a Lie algebra over k and consider

$$L' = k \oplus L$$

This is a coalgebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\Delta(1) = 1 \otimes 1$$

$$\epsilon(1) = 1$$

$$\epsilon(x) = 0$$

for $x \in L$.

Proposition.[CCES] The map $q : L' \otimes L' \to L'$ defined by

$$q((a, x) \otimes (b, y)) = (ab, bx + [x, y])$$

satisfies the self-distributive law.

More Solutions of the Yang–Baxter Equation

Let X be a coalgebra and $q : X \otimes X \to X$ a linear map. The linear map $B_q : X \otimes X \to X \otimes X$ defined by

$$B_q = (1 \otimes q)(\tau \otimes 1)(1 \otimes \Delta)$$

is said to be **induced from** q. Conversely, let $B : X \otimes X \to X \otimes X$ be a linear map. The linear map $q_B : X \otimes X \to X$ defined by

$$q_B : (\epsilon \otimes 1)B$$

is said to be **induced from** B.

Theorem.[CCES] Let X be a coalgebra with counit. Let $B : X \otimes X \to X \otimes X$ be a solution of the Yang–Baxter Equation. Suppose B also satisfies

$$(\epsilon \otimes \epsilon)B = \epsilon \otimes \epsilon$$

$$B_{q_B} = B$$

Then q_B satisfies the self-distributive law and is a coalgebra morphism.

Theorem.[CCES] Let X be a cocommutative coalgebra equipped with a linear map $q : X \otimes X \to X$ that preserves comultiplication and satisfies the self-distributive law. Then B_q is a solution of the Yang–Baxter Equation.

A **bialgebra** is an algebra A over a field k equipped with a **unit** map $\eta : k \to A$ and associative multiplication $m : A \otimes A \to A$ that is also a coalgebra such that the comultiplication and counit are algebra homomorphisms.

A **Hopf algebra** is a bialgebra C equipped with an **antipode** map $S : C \to C$ such that $m(S \otimes 1)\Delta = \eta\epsilon = m(1 \otimes S)\Delta$ where ϵ is the counit.

$$\begin{tikzcd}
 & C \otimes C \arrow[r, "S\otimes 1"] & C \otimes C \arrow[dr, "m"] & \\
C \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[r, "\eta"] & C \\
 & C \otimes C \arrow[r, "1\otimes S"] & C \otimes C \arrow[ur, "m"'] &
\end{tikzcd}$$

Examples:

- Let G be a group and k a field. The **group algebra** kG is a Hopf algebra with

$$\Delta(g) = g \otimes g$$

$$\epsilon(g) = 1$$

$$S(g) = g^{-1}$$

  for all $g \in G$

- Let G be a group and k a field and consider all functions on G with

$$\Delta(f)(x,y) = f(xy)$$

$$\epsilon(f) = f(e)$$

$$S(f)(x) = f(x^{-1})$$

  for $x, y \in G$, e the identity of G and $f : G \to k$.

Hopf Algebras and Self-Distributivity

Proposition.[CCES] Let H be a Hopf algebra. The map $q : H \otimes H \to H$ defined by

$$q = m(1 \otimes m)(S \otimes 1 \otimes 1)(\tau \otimes 1)(1 \otimes \Delta)$$

$$q(x \otimes y) = S(y_1)(xy_2)$$

satisfies the self-distributive law, where $\Delta(x) = x_1 \otimes x_2$.

Proposition.[CCES] Let H be a Hopf algebra equipped with a linear map $q : H \otimes H \to H$ that satisfies

$$q(q \otimes 1) = q(1 \otimes m)$$

$$(q\otimes m)(1\otimes\tau\otimes 1)(\Delta\otimes\Delta) = (1\otimes m)(\tau\otimes 1)(1\otimes\Delta)(1\otimes q)(\tau\otimes 1)(1\otimes\Delta)$$

Then B_q is a solution of the Yang–Baxter Equation.

What’s Next?

- What are more precise relationships among the Lie bracket, self-distributivity, solutions to the Yang-Baxter equation, Hopf algebras, and quantum groups?

- Can the cohomology theory for these generalized self-distributive structures be used to construct invariants of knots and knotted surfaces?

- What’s *really* going on here? What’s the conceptual meaning behind all this?

- How can these observations be extended to higher dimensions, such as to higher dimensional Lie algebras, or Lie 2-algebras?

- How, if at all, do the Zamolodchikov tetrahedron equation and the Jacobiator identity of a Lie 2-algebra relate to this self-distributive story?